



# Review for Calculus

## Rational Functions, Logarithms & Exponentials



### Definition and Domain of Rational Functions

A rational function is defined as the quotient of two polynomial functions.

$$F(x) = P(x) / Q(x)$$

The domain of F is the set of all real numbers except those for which  $Q(x) = 0$ .

Here are some examples of rational functions:

- $g(x) = (x^2 + 1) / (x - 1)$
- $h(x) = (2x + 1) / (x + 3)$

Example: Find the domain of each function given below.

- a.  $g(x) = (x - 1) / (x - 2)$
- b.  $h(x) = (x + 2) / x$

Solution

- a. For function g to be defined, the denominator  $x - 2$  must be different from zero or  $x$  not equal to 2.

Hence the domain of g is given by  $(-\infty, 2) \cup (2, +\infty)$

For function h to be defined, the denominator  $x$  must be different from zero or  $x$  not equal to 0. Hence the domain of h is given by

$$(-\infty, 0) \cup (0, +\infty)$$

### Vertical Asymptotes of Rational Functions

Let  $f(x) = 1/x$ .  $f(x)$  is not defined at  $x = 0$  (division by zero is not allowed). However what is the behavior of the graph "close" to zero?

In the tables below are values of function f as x approaches zero from the right ( $x > 0$ ) and as x approaches zero from the left ( $x < 0$ ).

x	1	0.1	0.01	0.001	0.000001
$f(x)$	1	10	100	1000	1000000

We note that as x approaches zero from the right,  $f(x)$  takes larger values. Is there a limit to the values of  $f(x)$ ? No,  $f(x)$  increases without bound.



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$x$	-1	-0.1	-0.01	-0.001	-0.000001
$f(x)$	-1	-10	-100	-1000	-1000000

We also note that as  $x$  approaches zero from the left,  $f(x)$  takes smaller values. Is there a limit to the values of  $f(x)$ ? No,  $f(x)$  decreases without bound. Therefore line  $x = 0$  is called the vertical asymptote and it is given by the zero of the denominator.

Let

$$f(x) = \frac{2}{x - 3}$$

The domain of  $f$  is the set of all real numbers except 3, since 3 makes the denominator zero and the division by zero is not allowed in mathematics. However we can try to find out how does the graph of  $f$  behave close to 3.

Let us evaluate function  $f$  at values of  $x$  close to 3 from the left. The values are shown in the table below:

X	1	2	2.5	2.8	2.9	2.99	2.999	2.99999
$f(x)$	-1	-2	-4	-10	-20	-200	-2000	$-2 * 10^5$

Let us now evaluate  $f$  at values of  $x$  close to 3 from the right.

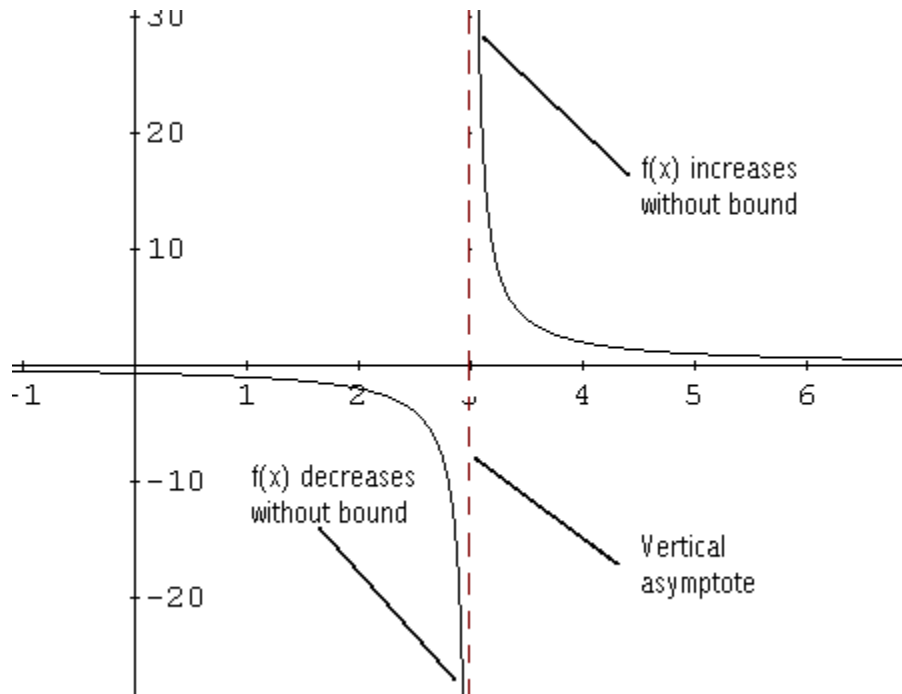
X	5	4	3.5	3.2	3.1	3.01	3.001	3.00001
$f(x)$	1	2	4	10	20	200	2000	$2 * 10^5$

The graph of  $f$  is shown below.



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### Notes

- 1) As  $x$  approaches 3 from the left or by values smaller than 3,  $f(x)$  decreases without bound.
- 2) As  $x$  approaches 3 from the right or by values larger than 3,  $f(x)$  increases without bound.

We say that the line  $x = 3$ , broken line, is the vertical asymptote for the graph of  $f$  (the value that makes the denominator equal to zero).

In general, the line  $x = a$  is a vertical asymptote for the graph of  $f$  if  $f(x)$  either increases or decreases without bound as  $x$  approaches  $a$  from the right or from the left. This is symbolically written as:

$$f(x) \rightarrow \infty \quad \text{or} \quad f(x) \rightarrow -\infty$$
$$\text{as } x \rightarrow a^+ \quad \text{or} \quad x \rightarrow a^-$$

### Horizontal Asymptotes of Rational Functions

Let  $f(x) = 1/x$ . What is the behavior of the graph of  $f$  as  $x$  becomes very large or very small?

The table below show values of  $f$  when  $x$  becomes very large, and when  $x$  becomes very small.



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x	1	10	100	1000	1000000
f(x)	1	0.1	0.01	0.001	0.000001
x	-1	-10	-100	-1000	-1000000
f(x)	-1	-0.1	-0.01	-0.001	-0.000001

As x takes smaller values or as x takes larger values, f(x) takes values close to zero and the graph approaches the line horizontal line  $y = 0$ . This line is called the horizontal asymptote.

Let

$$f(x) = \frac{2x + 1}{x}$$

1 Let x increase and find values of f(x).

x	1	10	10 <sup>3</sup>	10 <sup>6</sup>
f(x)	-3	2.1	2.001	2.000001

2 Let x decrease and find values of f(x).

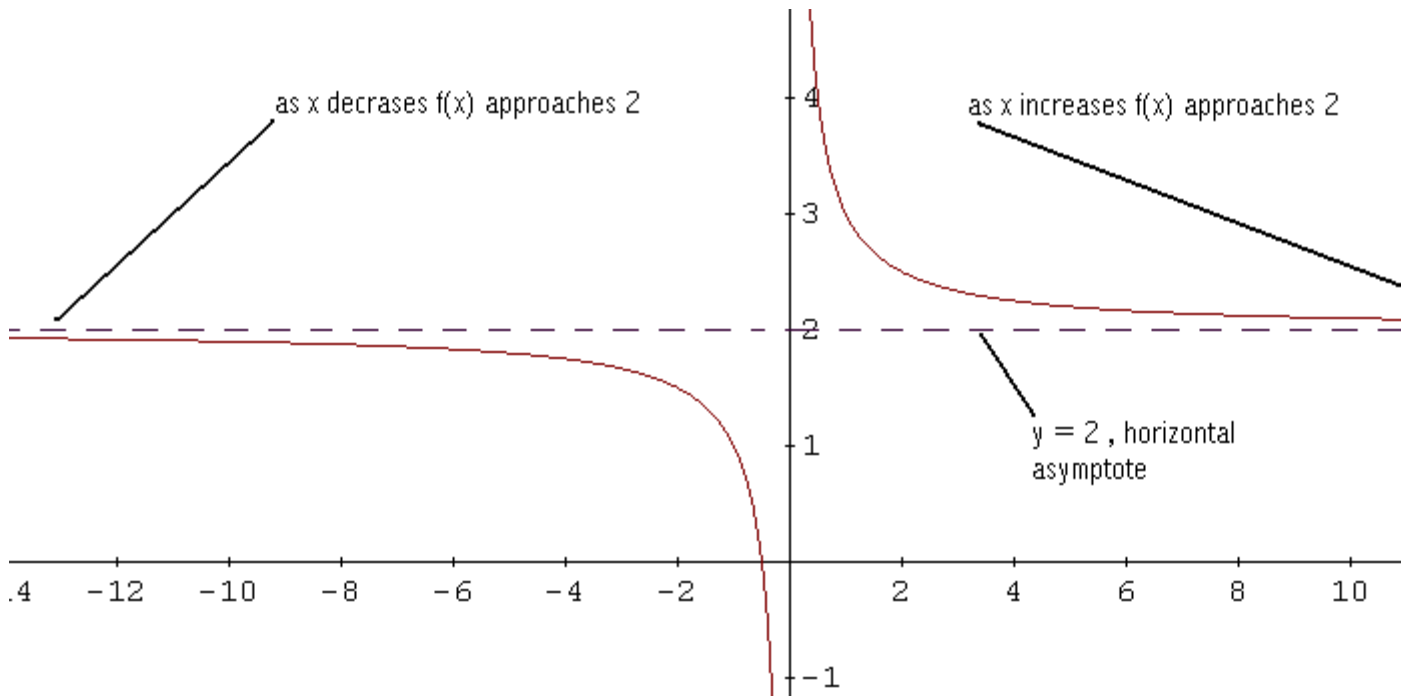
x	-1	-10	-10 <sup>3</sup>	-10 <sup>6</sup>
f(x)	1	1.9	1.999	1.999999

As x increases or decreases, the numerator is dominated by the term 2x and the numerator has only one term x. Therefore f(x) takes values close to  $2x / x = 2$ . See graphical behavior below.



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In general, the line  $y = b$  is a horizontal asymptote for the graph of  $f$  if  $f(x)$  approaches a constant  $b$  as  $x$  increases or decreases without bound.

### How to find the horizontal asymptote?

Let  $f$  be a rational function defined as follows

$$f(x) = \frac{a_m x^m + \dots + a_1 x + a_0}{b_n x^n + \dots + b_1 x + b_0}$$

### Theorem

$m$  is the degree of the polynomial in the numerator and  $n$  is the degree of the polynomial in the denominator.

case 1: For  $m < n$ , the horizontal asymptote is the line  $y = 0$ .

case 2: For  $m = n$ , the horizontal asymptote is the line  $y = a_m / b_n$

case 3: For  $m > n$ , there is no horizontal asymptote.



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**Example 1:** Let  $f$  be a rational function defined by

$$f(x) = \frac{x + 1}{x - 1}$$

- Find the domain of  $f$ .
- Find the  $x$  and  $y$  intercepts of the graph of  $f$ .
- Find the vertical and horizontal asymptotes for the graph of  $f$  if there are any.
- Use your answers to parts a, b and c above to sketch the graph of function  $f$ .

**Solution:**

a) The domain of  $f$  is the set of all real numbers except  $x = 1$ , since this value of  $x$  makes the denominator zero.

b) The  $x$  intercept is found by solving  $f(x) = 0$  or  $x + 1 = 0$ . The  $x$  intercept is at the point  $(-1, 0)$ .

The  $y$  intercept is at the point  $(0, f(0)) = (0, -1)$ .

c) The vertical asymptote is given by the zero of the denominator  $x = 1$ .

The degree of the numerator is 1 and the degree of the denominator is 1. They are equal and according to the theorem above, the horizontal asymptote is the line  $y = 1 / 1 = 1$

d) Although parts a, b and c give some important information about the graph of  $f$ , we still need to construct a sign table for function  $f$  in order to be able to sketch with ease.

The sign of  $f(x)$  changes at the zeros of the numerator and denominator. To find the sign table, we proceed as in solving rational inequalities. The zeros of the numerator and denominator which are  $-1$  and  $1$  divides the real number line into 3 intervals:

$$(-\infty, -1), (-1, 1), (1, +\infty)$$

We select a test value within each interval and find the sign of  $f(x)$ .

In  $(-\infty, -1)$ , select  $-2$  and find  $f(-2) = (-2 + 1) / (-2 - 1) = 1 / 3 > 0$ .

In  $(-1, 1)$ , select  $0$  and find  $f(0) = -1 < 0$ .

In  $(1, +\infty)$ , select  $2$  and find  $f(2) = (2 + 1) / (2 - 1) = 3 > 0$ .



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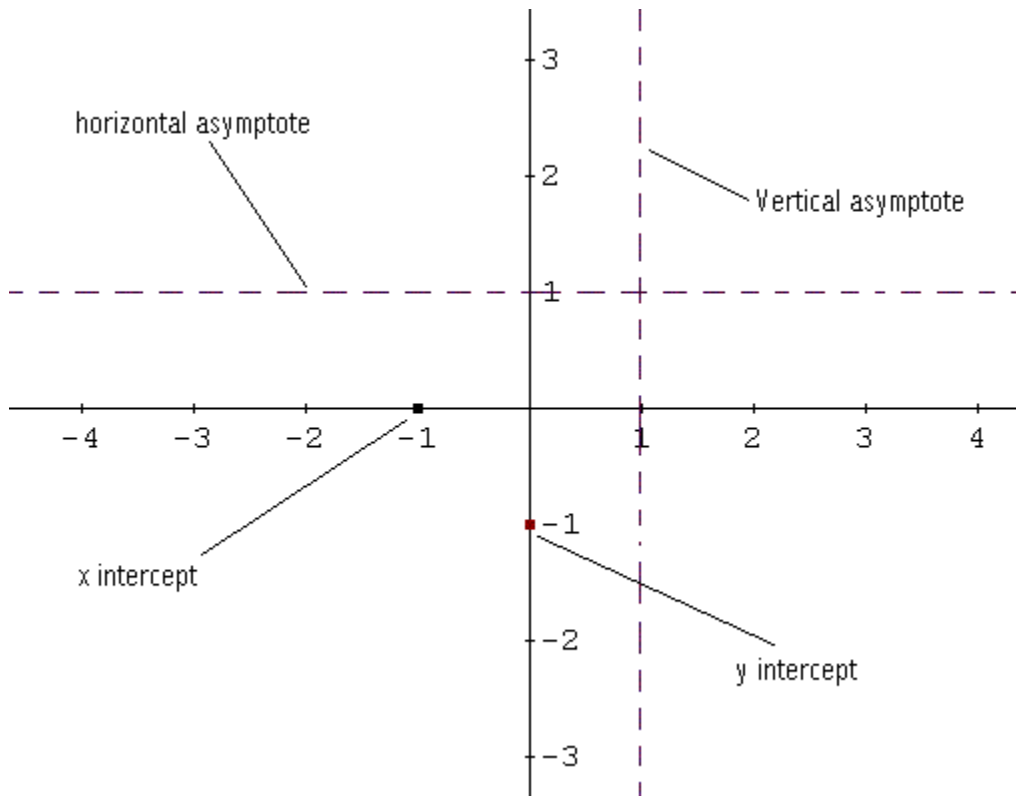


Let us put all the information about  $f$  in a table.

X	$-\infty$	-1		1	$+\infty$
f(x)	+	0 x- intercepts	-	V.A	+

In the table above V.A means vertical asymptote.

To sketch the graph of  $f$ , we start by sketching the x and y intercepts and the vertical and horizontal asymptotes in broken lines. See sketch below.



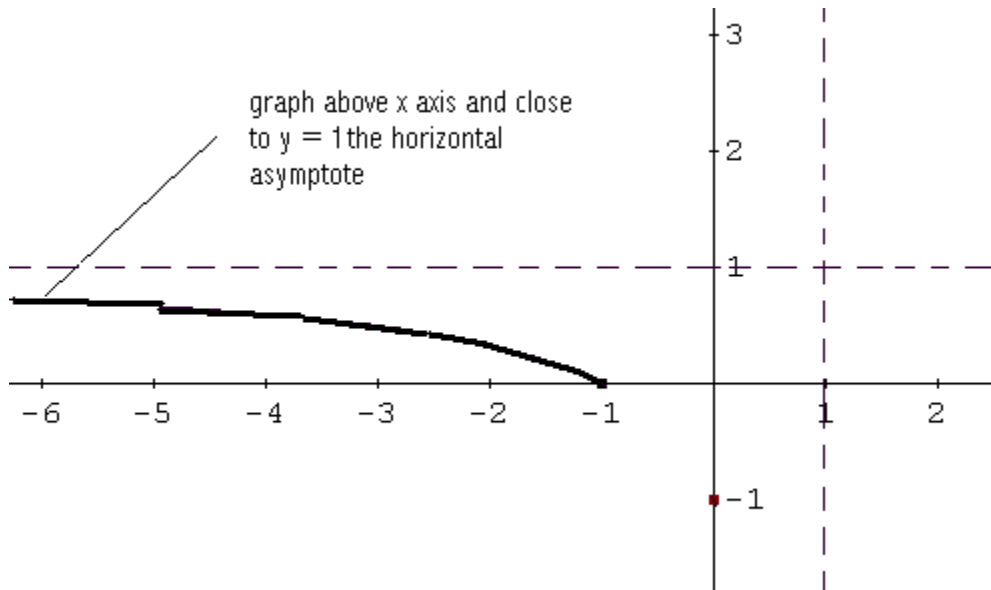
We now start sketching the graph of  $f$  starting from the left.

In the interval  $(-\infty, -1)$   $f(x)$  is positive hence the graph is above the x axis. Starting from left, we sketch  $f$  taking into account the fact that  $y = 1$  is a horizontal asymptote: The graph of  $f$  is close to this line on the left. See sketch below.

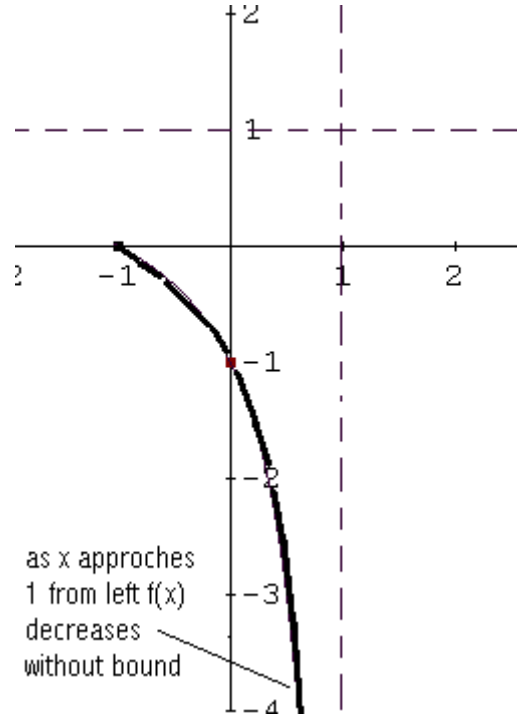


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Between  $(-1, 1)$   $f(x)$  is negative, hence the graph of  $f$  is below the  $x$  axis.  $(0, -1)$  is a  $y$  intercept and  $x = 1$  is a vertical asymptote: as  $x$  approaches 1 from left  $f(x)$  decreases without bound because  $f(x) < 0$  in  $(-1, 1)$ . See sketch below.

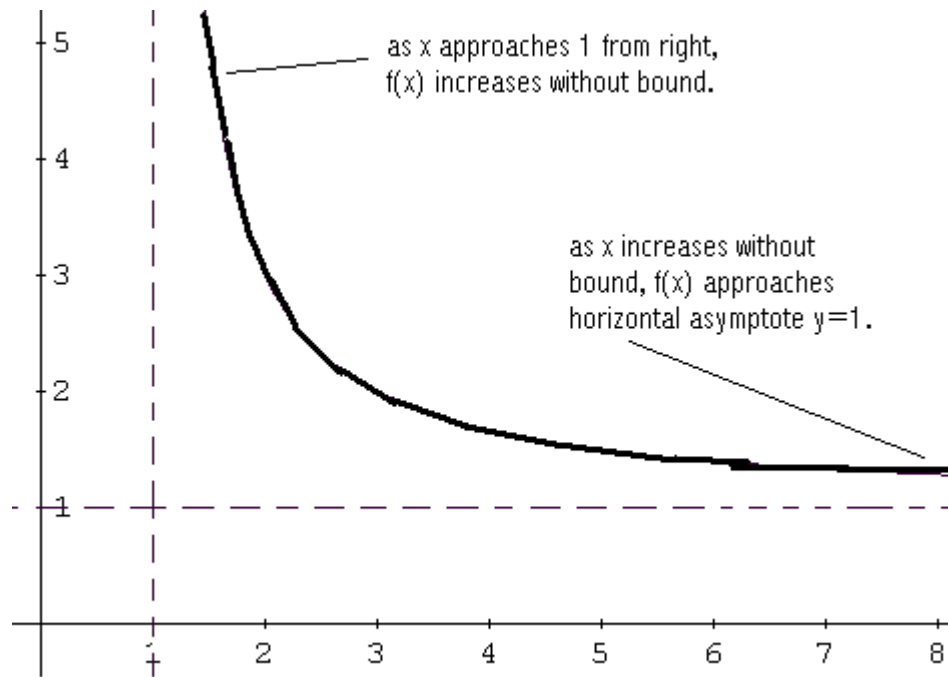


For  $(1, +\infty)$ ,  $f(x) > 0$  hence the graph is above the  $x$  axis. As  $x$  approaches 1 from the right, the graph of  $f$  increases without bound ( $f(x) > 0$ ). Also as  $x$  increases, the graph of  $f$  approaches  $y = 1$  the horizontal asymptote. See sketch below.

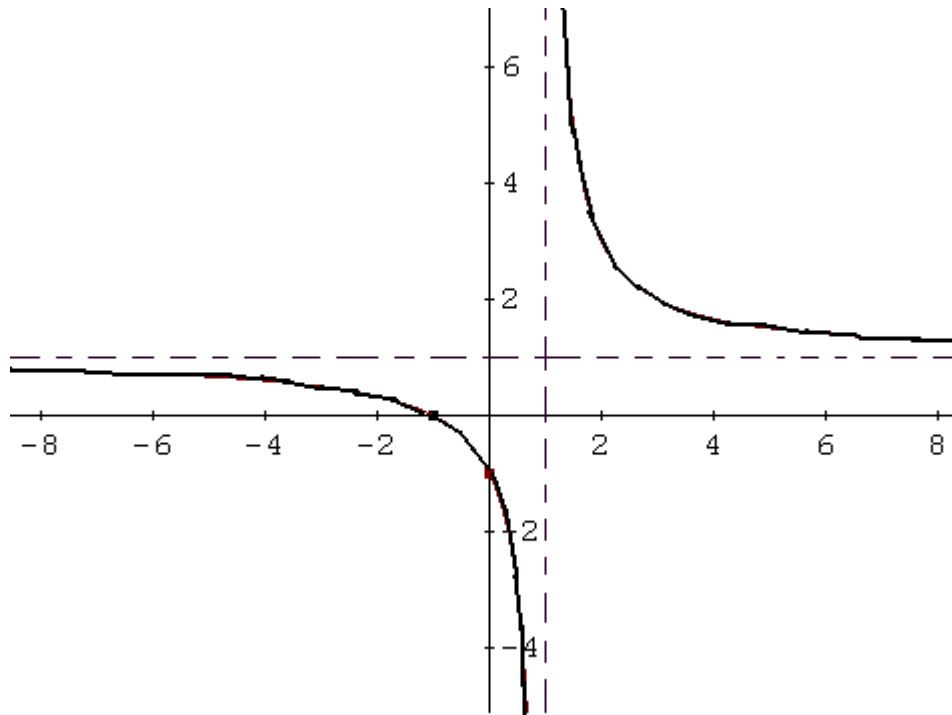


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We now put all "pieces" of the graph of  $f$  together to obtain the graph of  $f$ .





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**Try this Example** : Let  $f$  be a rational function defined by

$$f(x) = (-x + 2) / (x + 4)$$

- Find the domain of  $f$ .
- Find the  $x$  and  $y$  intercepts of the graph of  $f$ .
- Find the vertical and horizontal asymptotes for the graph of  $f$  if there are any.
- Use your answers to parts a, b and c above to sketch the graph of function  $f$ .

### **Find Equation Of Rational Function From Graph**

**Example1** : Find the equation of the rational function  $f$  of the form

$$f(x) = (x + a) / (bx + c)$$

whose graph has an  $x$  intercept at  $(2, 0)$ , a vertical asymptote at  $x = -1$  and a horizontal asymptote at  $y = 1/2$ .

### **Solution** :

The  $x$  intercept(s) is/are found by equating  $f(x) = 0$ , which implies the numerator should be zero. Furthermore, the numerator is equal to zero at  $x = 2$ .

$$0 = \frac{x + a}{bx + c} \Rightarrow x + a = 0$$

$$2 + a = 0$$

- Solve the above equation for  $a$ .  
 $a = -2$
- The horizontal asymptote is given by ratio of the leading coefficients in the numerator and denominator.  
 $1 / b = 1/2$
- Solve for  $b$   
 $b = 2$
- The vertical asymptote is given by the zero of the denominator. At  $x = -1$  the denominator in  $f(x)$  has to be equal to zero.  
 $-b + c = 0$
- Substitute 2 for  $b$  in the above equation  
 $-2 + c = 0$

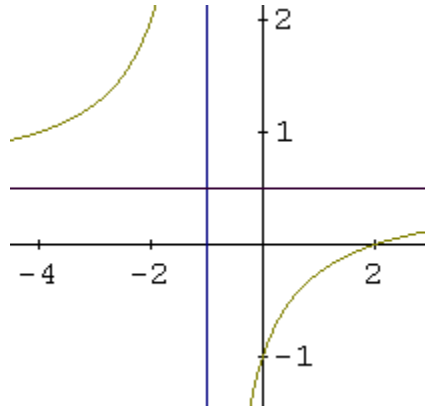


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- f. Solve the above equation for  $c$   
 $c = 2$
- g. The equation of the rational function is given by  
 $f(x) = (x - 2)/(2x + 2)$



**Try this Example** : Find the equation of the rational function  $f$  of the form

$$f(x) = (ax - 2) / (bx + c)$$

whose graph has an  $x$  intercept at  $(1, 0)$ , a vertical asymptote at  $x = -1$  and a horizontal asymptote at  $y = 2$ .



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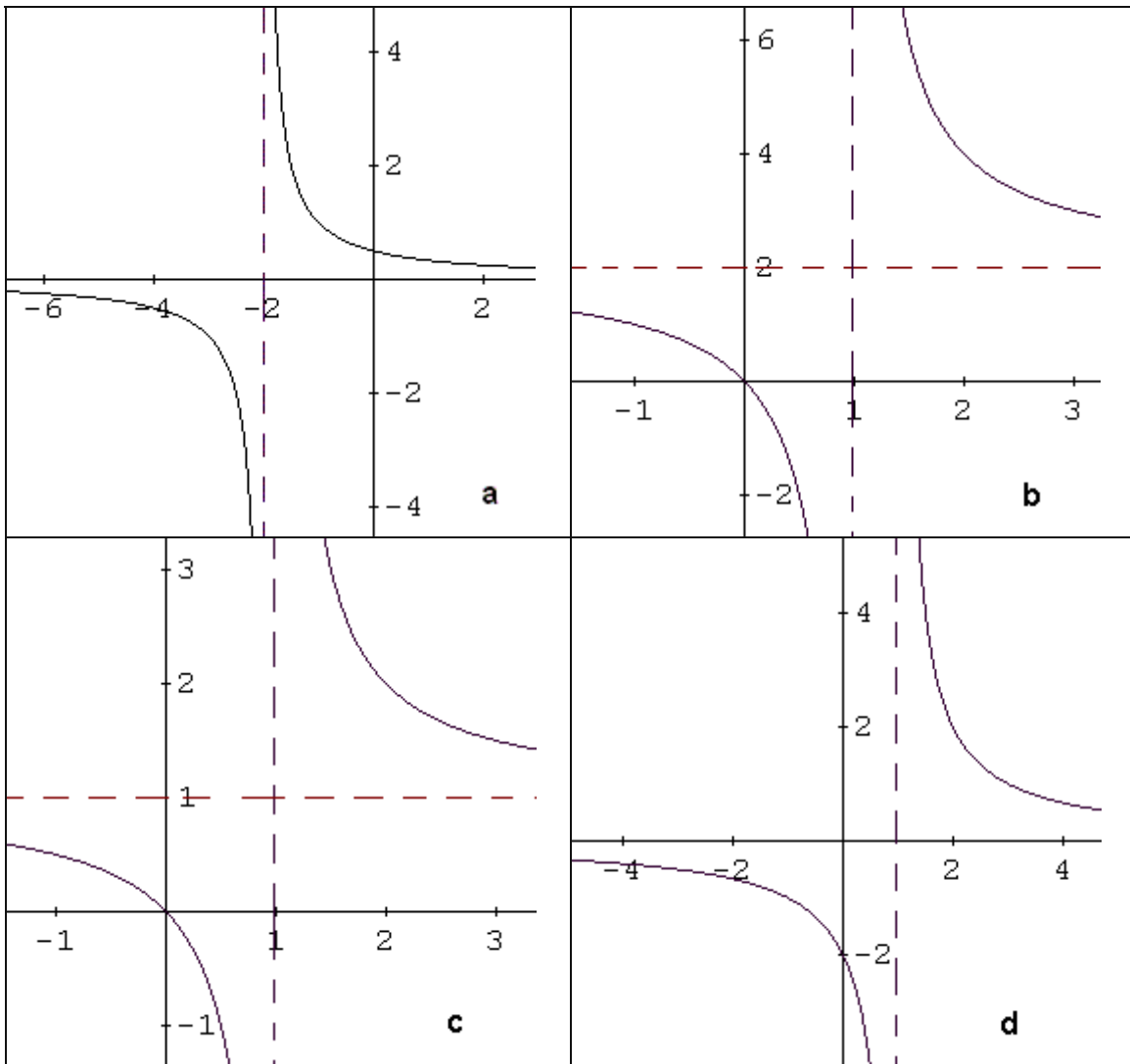
## Rational Functions, Logarithms & Exponentials



### Graphs of Rational Functions

This section is for identifying the graph of a rational function given its formula.

**Example 1** : Identify the graph of the function  $f(x) = 2x/(x - 1)$



### Solution:

The given function  $f(x) = 2x/(x - 1)$  has a vertical asymptote at  $x = 1$  and a horizontal asymptote at  $y = 2$ . If you examine the 4 graphs given above only graph b has a vertical asymptote at  $x = 1$  and a horizontal asymptote at  $y = 2$ .

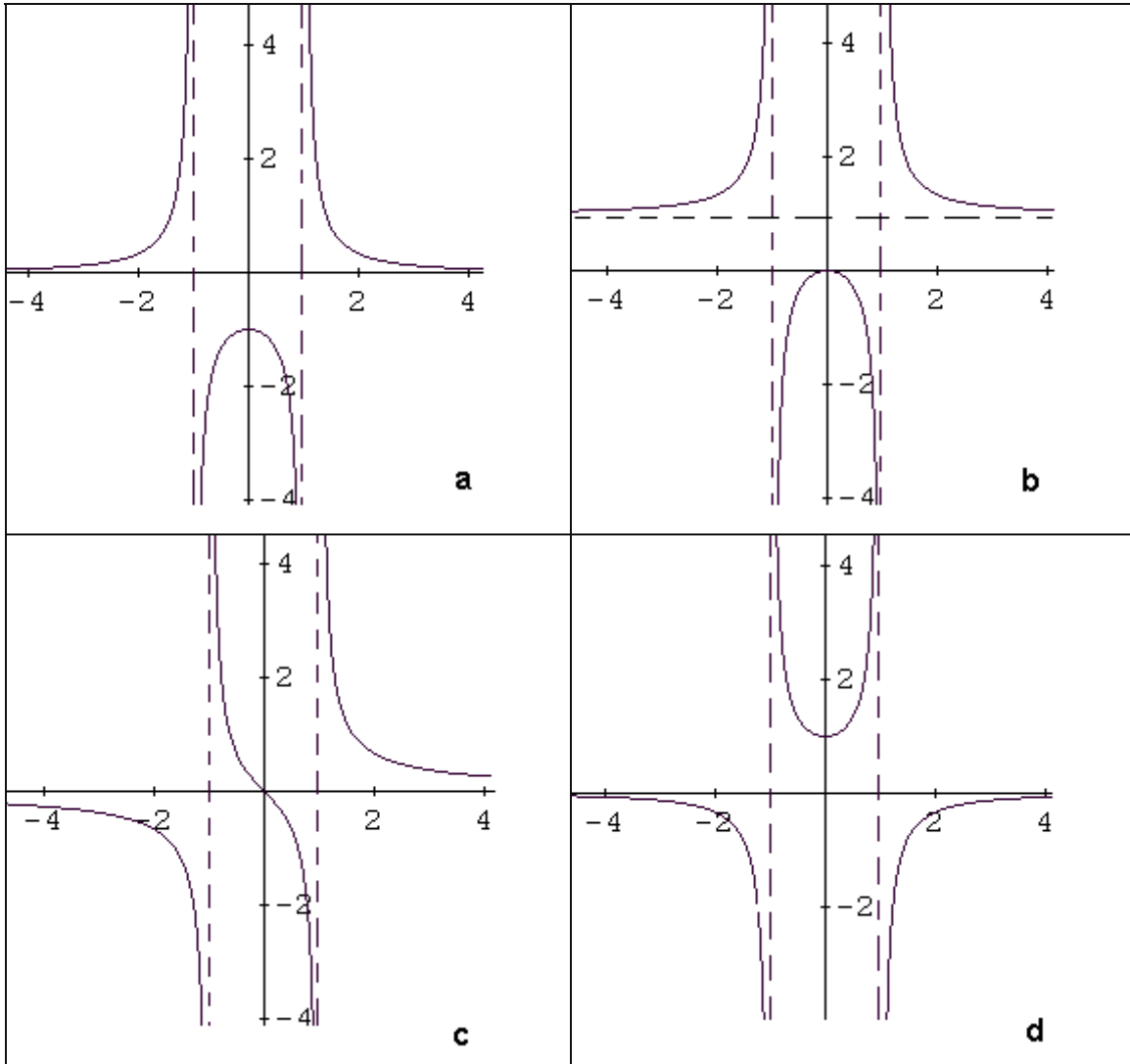


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**Example 2 :** Identify the graph of the function  $f(x) = x^2 / (x^2 - 1)$



**Solution:**

The given function  $f(x) = x^2 / (x^2 - 1)$  has 2 vertical asymptotes at  $x = 1$  and  $x = -1$  and a horizontal asymptote at  $y = 1$ . If you examine the 4 graphs given above only graph b has these characteristics.



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### Exponential Functions

If  $b$  is any number such that  $b > 0$  and  $b \neq 1$  then an exponential function is a function in the form,  

$$f(x) = b^x$$

where  $b$  is called the base and  $x$  can be any real number.  
 Notice that the  $x$  is now in the exponent and the base is a fixed number.

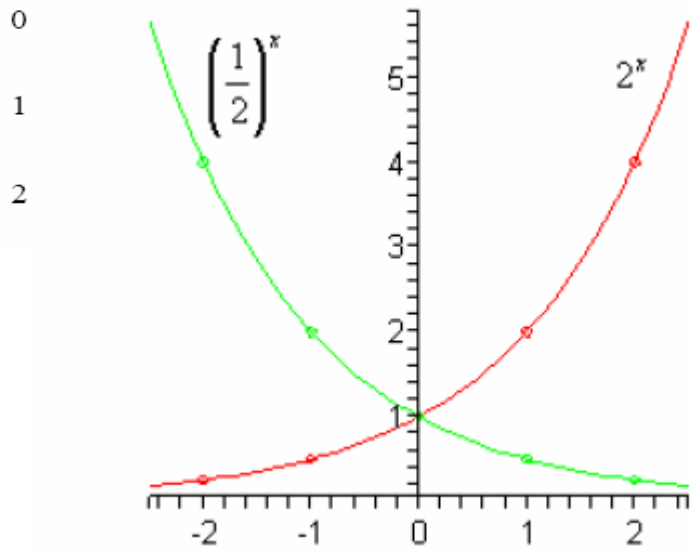
Example: Sketch the graph of  $f(x) = 2^x$  and  $g(x) = (1/2)^x$  on the same axis system.

**Solution**

Pick some value of  $x$  and do some function evaluation.

$x$	$f(x) = 2^x$	$g(x) = \left(\frac{1}{2}\right)^x$
-2	$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$g(-2) = \left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2 = 4$
-1	$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$g(-1) = \left(\frac{1}{2}\right)^{-1} = \left(\frac{2}{1}\right)^1 = 2$

Here is the sketch of the two graphs.



### Special exponential function: e

The special exponential function is the natural logarithm function  $e$ . Here it is,

$$f(x) = e^x$$

where  $e = 2.718281827\dots$ . Note the difference between  $f(x) = b^x$  and  $f(x) = e^x$ . In the first case  $b$  is any number that meets the restrictions given above while  $e$  is a very specific number. Also note that  $e$  is an irrational number like  $\pi$ .

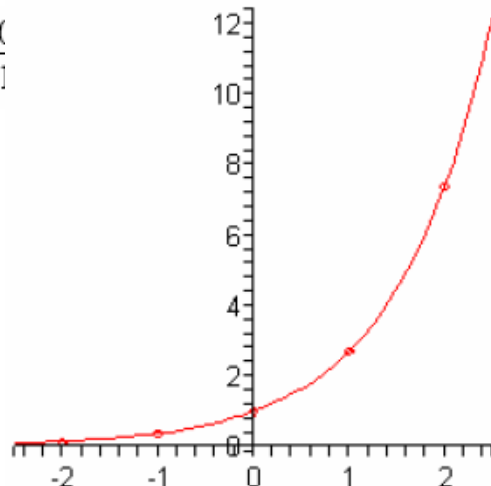
**Example:** Sketch the graph of  $f(x) = e^x$ .

**Solution:** Let's first build up a table of values for this function.

$x$	-2	-1	0
$f(x)$	0.1353...	0.3679...	1

To get these evaluation (with the exception of  $x = 0$ ) you will need to use a calculator. Make sure that you can use your calculator and verify these numbers.

Here is a sketch of this graph.





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Notice that this is an increasing graph as we should expect since  $e = 2.718281827 > 1 \dots$

**Try this:**  $g(x) = 5e^{1-x} - 4$

### Exponential Growth & Decay

In many applications, a quantity  $N$  grows or decays, according to the function  $N(t) = N_0 e^{kt}$ . In this function,  $N$  is a function of time  $t$ , and  $N_0$  is the value of  $N$  at time  $t=0$ . If  $k$  is a positive constant, then  $N(t) = N_0 e^{kt}$  is called an exponential growth function. If  $k$  is a negative constant, then  $N(t) = N_0 e^{kt}$  is called an exponential decay function.

### Example:

Find the exponential growth function for a town whose population was 16,400 in 1990 and 20,200 in 2000.

### Solution

We need to determine  $N_0$  and  $k$  in  $N(t) = N_0 e^{kt}$ . If we represent the year 1990 by  $t=0$ , then our given data are  $N(0)=16,400$  and  $N(10)=20,200$ . Because  $N_0$  is defined to be  $N(0)$ , we know that  $N_0=16,400$ . To determine  $k$ , substitute  $t=10$  and  $N_0=16,400$  in  $N(t) = N_0 e^{kt}$  to produce

$$N(10)=16,400 e^{k \cdot 10}$$

$$20,200 = 16,400 e^{10k}$$

$$\frac{20,200}{16,400} = e^{10k}$$

$$\ln \frac{20,200}{16,400} = 10k$$

$$\frac{1}{10} \ln \frac{20,200}{16,400} = k$$

$$0.0208 \approx k$$

The exponential growth function is  $N(t) \approx 16,400 e^{0.0208 t}$

### Logistic Model

The population growth function is given by  $P(t) = P_0 e^{kt}$ . The logistic model is a restricted growth model that takes into consideration the effects of limited resources.

The population at time  $t \geq 0$  is given by

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

Where  $c$  is the carrying capacity (the maximum population that can be supported by available resources as  $t \rightarrow \infty$ ) and  $b$  is a positive constant called the Growth Rate Constant.



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The initial population is  $P_0 = P(0)$ . To find the constant  $a$ , use the formula

$$a = \frac{c - P_0}{P_0}$$

**Example:**

At the beginning of 2002, the coyote population in the wilderness area was estimated at 200. By the beginning of 2004, the population had increased to 250. A park ranger estimates that the carrying capacity of the wilderness area is 500 coyotes.

Use the given data to determine the growth rate constant,  $b$ , for the logistic model of this coyote population.

**Solution:**

If we represent the beginning of the year 2002 by  $t=0$ , then the beginning of the year 2004 will be represented by  $t=2$ . In the logistic model, make the following substitutions:  $P(2)=250$ ,  $c=500$  and

$$a = \frac{c - P_0}{P_0} = \frac{500 - 200}{200} = 1.5$$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

$$P(2) = \frac{500}{1 + 1.5e^{-b \cdot 2}}$$

$$250 = \frac{500}{1 + 1.5e^{-b \cdot 2}}$$

$$250(1 + e^{-b \cdot 2}) = 500$$

$$1 + 1.5e^{-b \cdot 2} = \frac{500}{250}$$

$$1.5e^{-b \cdot 2} = 1$$

$$e^{-b \cdot 2} = \frac{1}{1.5}$$

$$-2b = \ln\left(\frac{1}{1.5}\right)$$



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$$b \approx 0.2027$$

$$P(t) = \frac{500}{1 + 1.5e^{-0.2027t}}$$

### Logarithms

Logarithmic functions are the inverse of exponential functions.

For example, the inverse of  $y = a^x$  is  $y = \log_a x$ , which is the same as  $x = a^y$ .

Note: Logarithms written without a base are understood to be base 10.

This definition is explained by knowing how to convert exponential equations to logarithmic form, and logarithmic equations to exponential form.

**Examples:** 1. Convert to logarithmic form:  $8 = 2^x$

Solution: Remember that the logarithm is the exponent.

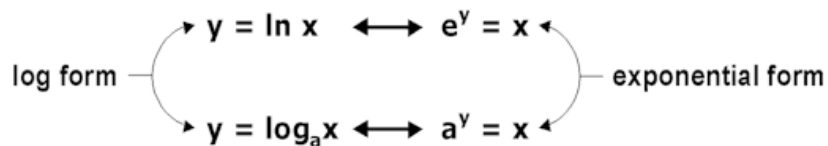
$$x = \log_2 8$$

2. Convert to exponential form:  $y = \log_3 5$

Solution: Remember that the logarithm is the exponent.

$$3^y = 5$$

How to convert from exponential to logarithmic form and from logarithmic to exponential form?



### Properties of Logarithms

For any positive numbers  $x$  and  $y$ ,  $\log_a (x * y) = \log_a x + \log_a y$  when  $a \neq 1$ .

$$\log_a (x/y) = \log_a x - \log_a y.$$

Example: Simplify:  $\log_2 x + \log_2 6$ .

$$\text{Solution: } \log_2 (x * 6) = \log_2 6x$$

For any positive numbers  $x$  and  $p$ ,  $\log_a x^p = p \log_a x$ .

Example: Simplify:  $\log_b 9^{-x}$

$$\text{Solution: } -x \log_b 9$$

$$\log_a a = 1 \quad \& \quad \log_a 1 = 0$$