Square Roots and Other Radicals

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Radicals - Definition

Radicals, or roots, are the opposite operation of applying exponents. A power can be undone with a radical and a radical can be undone with a power. For example, if you square 2, you get 4, and if you take the square root of 4, you get 2; if you square 3, you get 9, and if you take the square root of 9, you get 3:

$$2^2 = 4, \text{ so } \sqrt{4} = 2$$
$$3^2 = 9, \text{ so } \sqrt{9} = 3$$

The $\sqrt{\ }$ symbol is called the radical symbol. Technically, just the check mark part of the symbol is the radical; the line across the top is called the vinculum. The expression $\sqrt{9}$ is read as "root nine", "radical nine", or "the square root of nine".

Numbers can be raised to powers other than just 2; you can cube things, raise them to the fourth power, raise them to the 100th power, and so forth. In the same way, you can take the cube root of a number, the fourth root, the 100th root, and so forth. To indicate some root other than a square root, you use the same radical symbol, but you insert a number into the radical, tucking it into the check mark part. For example:

$$4^3 = 64, \text{ so } \sqrt[3]{64} = 4$$

The "3" inside the check mark part is the "index" of the radical. The "64" is "the argument of the radical", also called "the radicand". Since most radicals you see are square roots, the index is not included on square roots. While " $\sqrt[2]{\ }$ " would be technically correct, it is not used.

Common Radicals:

a square (second) root is written as $\sqrt{\ }$

a cube (third) root is written as $\sqrt[3]{\ }$

a fourth root is written as $\sqrt[4]{\ }$

a fifth root is written as: $\sqrt[5]{\ }$

You can take any counting number, square it, and end up with a nice neat number. But the process doesn't always work going backwards. For instance, consider $\sqrt{3}$, the square root of three. There is no nice neat number that squares to 3, so $\sqrt{3}$ cannot be simplified as a nice whole number. You can deal with $\sqrt{3}$ in either of two ways: If you are doing a word problem and are trying to find, say, the rate of speed, then you would grab your calculator and find the decimal approximation of $\sqrt{3}$:

$$\sqrt{3} \approx 1.732050808$$

Then you'd round the above value to an appropriate number of decimal places and use a real-world unit or label, like "1.7 ft/sec". On the other hand, you may be solving a plain old math exercise, something with no "practical" application. Then they would almost certainly want the "exact" value, so you'd give your answer as being simply $\sqrt{3}$.

Simplifying Square Roots

To simplify a square root, you take out anything that is a "perfect square". That is, you take out front anything that has two copies of the same factor:

$$\sqrt{4} = \sqrt{2^2} = 2$$

$$\sqrt{49} = \sqrt{7^2} = 7$$

$$\sqrt{225} = \sqrt{15^2} = 15$$

Sometimes the argument of a radical is not a perfect square, but it may contain a square amongst its factors. To simplify, factor the argument and take out anything that is a square. This means finding pairs inside the radical and moving it out front. To do this, use the fact that you can switch between the multiplication of roots and the root of a multiplication. In other words, radicals can be manipulated similarly to powers:

$$(ab)^n = a^n b^n \quad \text{and} \quad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

Examples Using Simplification of Square Roots

- **Simplify $\sqrt{144}$**

  There are various ways to approach this simplification. One would be by factoring and then taking two different square roots:

  $$\sqrt{144} = \sqrt{9 \times 16} = \sqrt{9}\sqrt{16} = 3 \times 4 = 12$$

  The square root of 144 is **12**.

Another way to approach this simplification is if you already knew that $12^2 = 144$, so the square root of 144 must be 12. However, using the steps above, it is easier to see how to switch back and forth between the different formats (multiplication inside one radical, versus multiplication of two radicals) to help in the simplification process.

- **Simplify $\sqrt{24}\sqrt{6}$**

  Neither 24 nor 6 is a perfect square, so simplify by putting them under one radical and multiplying them together.

  $$\sqrt{24}\sqrt{6} = \sqrt{24 \times 6} = \sqrt{144} = \sqrt{12 \times 12} = \mathbf{12}$$

- **Simplify $\sqrt{75}$**

  $$\sqrt{75} = \sqrt{3 \times 25} = \sqrt{25}\sqrt{3} = \mathbf{5\sqrt{3}}$$

This answer is pronounced as "five, root three". It is proper form to put the radical at the end of the expression. Not only is "$\sqrt{3}5$" non-standard, it is very hard to read, especially when hand-written.

Write neatly, because "$5\sqrt{3}$" is not the same as "$\sqrt[5]{3}$".

- **Simplify $\sqrt{72}$**

  Since 72 factors as 2×36, and since 36 is a perfect square, then:

  $$\sqrt{72} = \sqrt{2 \times 36} = \sqrt{2 \times \underline{6 \times 6}} = \mathbf{6\sqrt{2}}$$

Since there was only one copy of the factor 2 in the factorization 2×6×6, the left-over 2 cannot come out of the radical and has to be left behind.

- **Simplify $\sqrt{4500}$**

$$\begin{aligned}\sqrt{4500} &= \sqrt{45\times 100} = \sqrt{5\times 9\times 100}\\ &= 3\times 10\times\sqrt{5} = \mathbf{30\sqrt{5}}\end{aligned}$$

Variables in a radical's argument are simplified in the same way: whatever you've got a pair of can be taken "out front".

- **Simplify $\sqrt{16x^4}$**

$$\sqrt{16x^4} = \sqrt{\underline{4\times 4}\times\underline{xx\times xx}} = 4xx = \mathbf{4x^2}$$

- **Simplify $\sqrt{12a^4b^7c^3}$**

  The 12 is the product of 3 and 4, so I have a pair of 2's but a 3 left over. Also, I have two pairs of *a*'s; three pairs of *b*'s, with one *b* left over; and one pair of *c*'s, with one *c* left over. So the root simplifies as:

$$\begin{aligned}\sqrt{12a^4b^7c^3} &= \sqrt{3\times\underline{2\times 2}\times\underline{aa\times aa}\times\underline{bbb\times bbb}\times b\times\underline{c\times c}\times c}\\ &= 2\times aa\times bbb\times c\times\sqrt{3\times b\times c}\\ &= \mathbf{2a^2b^3c\sqrt{3bc}}\end{aligned}$$

For radical expressions, any variables outside the radical should go in front of the radical, as shown above.

- **Simplify $\sqrt{20r^{18}st^{21}}$**

  Just use what you know about powers. The 20 factors as 4×5, with the 4 being a perfect square. The r^{18} has nine pairs of *r*'s; the *s* is unpaired; and the t^{21} has ten pairs of *t*'s, with one *t* left over. Then:

$$\begin{aligned}\sqrt{20r^{18}st^{21}} &= \sqrt{4\times r^{18}\times t^{20}\times 5\times s\times t}\\ &= \mathbf{2r^9t^{10}\sqrt{5st}}\end{aligned}$$

Multiplying Square Roots

In order to multiply roots, they must first be simplified to make the process easier.

Simplifying multiplied radicals is pretty simple. Use the fact that the product of two radicals is the same as the radical of the product, and vice versa.

- **Write as the product of two radicals:** $\sqrt{6}$

$$\sqrt{6} = \sqrt{2 \times 3}$$
$$= \mathbf{\sqrt{2}\sqrt{3}}$$

- **Simplify by writing with no more than one radical:** $\sqrt{2}\sqrt{8}$

$$\sqrt{2}\sqrt{8} = \sqrt{2 \times 8} = \sqrt{16}$$
$$= \sqrt{4 \times 4} = \mathbf{4}$$

- **Simplify by writing with no more than one radical:** $\sqrt{3}\sqrt{6}$

$$\sqrt{3}\sqrt{6} = \sqrt{3 \times 6} = \sqrt{\underline{3 \times 3} \times 2} = \mathbf{3\sqrt{2}}$$

- **Simplify by writing with no more than one radical:** $\sqrt{6}\sqrt{15}\sqrt{10}$

$$\sqrt{6}\sqrt{15}\sqrt{10} = \sqrt{6 \times 15 \times 10}$$
$$= \sqrt{2 \times 3 \times 3 \times 5 \times 5 \times 2}$$
$$= \sqrt{\underline{2 \times 2} \times \underline{3 \times 3} \times \underline{5 \times 5}}$$
$$= 2 \times 3 \times 5$$
$$= \mathbf{30}$$

- **Simplify by writing with no more than one radical:** $\sqrt{4x}\sqrt{5x^3}$

$$\sqrt{4x}\sqrt{5x^3} = \sqrt{4 \times 5 \times x \times xxx}$$
$$= \sqrt{\underline{2 \times 2} \times 5 \times \underline{xx} \times \underline{xx}}$$
$$= 2 \times xx \times \sqrt{5} = \mathbf{2x^2\sqrt{5}}$$

Adding and Subtracting Square Roots using Simplification

Just as with regular numbers, square roots can be added together. However, you might not be able to simplify the addition all the way down to one number.

A good analogy to remember is: you can't add apples and oranges, so you also cannot combine unlike radicals. To add radical terms together, they must have the same radical part.

- **Simplify:** $\mathbf{2\sqrt{3}+3\sqrt{3}}$

  Since the radical is the same in each term (the square root of three), combine the terms. The problem states that we have two copies of the radical, added to another three copies. This gives five copies:

$$\begin{aligned} 2\sqrt{3}+3\sqrt{3} &= (2+3)\sqrt{3} \\ &= \mathbf{5\sqrt{3}} \end{aligned}$$

That middle step, with the parentheses, shows the reasoning that justifies the final answer. You probably won't ever need to show this step, but it's what should be going through your mind.

- **Simplify:** $\mathbf{\sqrt{3}+4\sqrt{3}}$

  The radical part is the same in each term, so perform the addition. To help keep track that the first term means one copy of the square root of three, just insert the understood 1:

$$\begin{aligned} \sqrt{3}+4\sqrt{3} &= 1\sqrt{3}+4\sqrt{3} \\ &= (1+4)\sqrt{3} \\ &= \mathbf{5\sqrt{3}} \end{aligned}$$

Don't assume that expressions with unlike radicals cannot be simplified. It is possible that, *after simplifying the radicals*, the expression can indeed be simplified.

- **Simplify:** $\mathbf{\sqrt{9}+\sqrt{25}}$

  To simplify a radical addition, first see if each radical term can be simplified. In this particular case, the square roots simplify completely down to whole numbers:

$$\sqrt{9}+\sqrt{25} = 3+5 = \mathbf{8}$$

- **Simplify:** $\mathbf{3\sqrt{4}+2\sqrt{4}}$

  Here, we have three copies of the radical, plus another two copies, giving— Wait a minute! Those radicals can be simplified right down to whole numbers:

  $$3\sqrt{4}+2\sqrt{4}=3\times 2+2\times 2=6+4=\mathbf{10}$$

Don't worry if you don't see a simplification right away. If you did not notice until the end that the radical simplified, your steps would have been different, but the final answer would have been the same:

$$3\sqrt{4}+2\sqrt{4}=5\sqrt{4}=5\times 2=10$$

- **Simplify:** $\mathbf{3\sqrt{3}+2\sqrt{5}+\sqrt{3}}$

  Since we can only combine the like radicals, we end up with two terms in the answer:

  $$\begin{aligned}3\sqrt{3}+2\sqrt{5}+\sqrt{3}&=3\sqrt{3}+1\sqrt{3}+2\sqrt{5}\\&=\mathbf{4\sqrt{3}+2\sqrt{5}}\end{aligned}$$

- **Simplify:** $\mathbf{3\sqrt{8}+5\sqrt{2}}$

  Remember: Radicals can only be added together if they have the same radical part. Try to simplify the radical in the first term to create like terms:

  $$\begin{aligned}3\sqrt{8}+5\sqrt{2}&=3\sqrt{\underline{2\times 2}\times 2}+5\sqrt{2}\\&=3\times 2\sqrt{2}+5\sqrt{2}\\&=6\sqrt{2}+5\sqrt{2}=\mathbf{11\sqrt{2}}\end{aligned}$$

- **Simplify:** $\sqrt{18}-2\sqrt{27}+3\sqrt{3}-6\sqrt{8}$

  Simplify as many of the radicals as possible:

$$\begin{aligned}
&\sqrt{18}-2\sqrt{27}+3\sqrt{3}-6\sqrt{8}\\
&\quad=\sqrt{3\times3\times2}-2\sqrt{3\times3\times3}+3\sqrt{3}-6\sqrt{2\times2\times2}\\
&\quad=3\sqrt{2}-2\times3\sqrt{3}+3\sqrt{3}-6\times2\sqrt{2}\\
&\quad=3\sqrt{2}-6\sqrt{3}+3\sqrt{3}-12\sqrt{2}\\
&\quad=3\sqrt{2}-12\sqrt{2}-6\sqrt{3}+3\sqrt{3}\\
&\quad=\mathbf{-9\sqrt{2}-3\sqrt{3}}
\end{aligned}$$

- **Simplify:** $2\sqrt{3}+3\sqrt{5}$

  These two terms have unlike radical parts, and neither radical can be simplified. The expression $2\sqrt{3}+3\sqrt{5}$ cannot be simplified any further and the answer must be:

  $\mathbf{2\sqrt{3}+3\sqrt{5}}$
  (The expression is already fully simplified)

Adding and Subtracting Square Roots using Expansion

Expansion of square roots involves multiplying and then simplification.

- **Expand:** $\sqrt{2}\left(3+\sqrt{3}\right)$

  First, distribute the square root of two across the parentheses:

$$\begin{aligned}
\sqrt{2}\left(3+\sqrt{3}\right)&=\sqrt{2}(3)+\sqrt{2}\left(\sqrt{3}\right)\\
&=3\sqrt{2}+\sqrt{2\times3}\\
&=\mathbf{3\sqrt{2}+\sqrt{6}}
\end{aligned}$$

This simplification involved turning a product of radicals into one radical containing the value of the product (being 2×3 = 6). Radicals may need to be manipulated in both directions in order to properly simplify.

- **Expand:** $\sqrt{3}\left(2\sqrt{3}+\sqrt{5}\right)$

$$\begin{aligned}\sqrt{3}\left(2\sqrt{3}+\sqrt{5}\right) &= \sqrt{3}\left(2\sqrt{3}\right)+\sqrt{3}\left(\sqrt{5}\right)\\ &= 2\sqrt{3\cdot 3}+\sqrt{3\cdot 5}\\ &= 2\cdot 3+\sqrt{15}\\ &= \mathbf{6+\sqrt{15}}\end{aligned}$$

- **Expand:** $\left(1+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

  Multiply using the FOIL method:

First: (1)*(3) = 3

Outer: (1)*(-$\sqrt{2}$) = -1$\sqrt{2}$

Inner: ($\sqrt{2}$)*(3) = 3$\sqrt{2}$

Last: ($\sqrt{2}$)*(-$\sqrt{2}$) = -$\sqrt{2*2}$

Answer: (1 + $\sqrt{2}$)(3 - $\sqrt{2}$) = 3-1$\sqrt{2}$ +3$\sqrt{2}$ - $\sqrt{2*2}$ = 3 + 2$\sqrt{2}$ - 2 = $\boxed{1+2\sqrt{2}}$

- **Simplify** $\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-\sqrt{6}\right)$

  Multiply using the FOIL method:

First: ($\sqrt{3}$)*($\sqrt{3}$) = $\sqrt{3*3}$

Outer: ($\sqrt{3}$)*(-$\sqrt{6}$) = -$\sqrt{6*3}$

Inner: ($\sqrt{5}$)*($\sqrt{3}$) = $\sqrt{5*3}$

Last: ($\sqrt{5}$)*(-$\sqrt{6}$) = -$\sqrt{5*6}$

Answer: ($\sqrt{3}+\sqrt{5}$)($\sqrt{3}-\sqrt{6}$) = $\sqrt{3*3}$ - $\sqrt{6*3}$ + $\sqrt{5*3}$ - $\sqrt{5*6}$ =
$\sqrt{9}-\sqrt{18}+\sqrt{15}-\sqrt{30}=\boxed{3+\sqrt{15}-3\sqrt{2}-\sqrt{30}}$

Multiplying Conjugates

- **Simplify:** $\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-\sqrt{5}\right)$

  Multiply using the FOIL method:

First: $(\sqrt{3})*(\sqrt{3}) = \sqrt{3*3}$

Outer: $(\sqrt{3})*(-\sqrt{5}) = -\sqrt{3*5}$

Inner: $(\sqrt{5})*(\sqrt{3}) = \sqrt{5*3}$

Last: $(\sqrt{5})*(-\sqrt{5}) = -\sqrt{5*5}$

Answer: $(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5}) = \sqrt{3*3} - \sqrt{3*5} + \sqrt{5*3} - \sqrt{5*5} =$
$\sqrt{9}-\sqrt{15}+\sqrt{15}-\sqrt{25} = 3-5 = \boxed{-2}$

Note in the last example how the answer contained only integers and the terms do *not* include any radicals. When multiplying two radical binomials together, the answer can contain no radicals. One example of when this can occur is when the two binomials are the same except for the sign in the middle: one has a plus and the other has a minus. This pair of binomials, with the second factor differing only in the one sign in the middle, is very important; in fact, this "same except for the sign in the middle" second factor has its own name:

Given the radical expression $\sqrt{a}+\sqrt{b}$, the **conjugate** is the expression $\sqrt{a}-\sqrt{b}$.

The conjugate has the same numbers but the opposite sign in the middle. So not only is $\sqrt{a}-\sqrt{b}$ the conjugate of $\sqrt{a}+\sqrt{b}$, but $\sqrt{a}+\sqrt{b}$is the conjugate of $\sqrt{a}-\sqrt{b}$.

When you multiply conjugates, the middle term (*ab*) will cancel out:

$$(\sqrt{a}+\sqrt{b})*(\sqrt{a}-\sqrt{b}) = \sqrt{a*a}+\sqrt{ab}-\sqrt{ab}-\sqrt{b*b} = a-b$$

This is an important concept, especially when dealing with fractions containing radicals in their denominators.

<u>Dividing by Square Roots</u>

Just as you can swap between the multiplication of radicals and a radical containing a multiplication, you can also swap between the division of roots and one root containing a division.

- **Simplify:** $\sqrt{\frac{8}{2}}$

  Simplify this by working inside, and then taking the square root:

  $$\sqrt{\frac{8}{2}} = \sqrt{4} = \mathbf{2}$$

  ...or by splitting the division into two radicals, simplifying, and cancelling:

  $$\sqrt{\frac{8}{2}} = \frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = \mathbf{2}$$

- **Simplify:** $\sqrt{\frac{3}{25}}$

  $$\sqrt{\frac{3}{25}} = \frac{\sqrt{3}}{\sqrt{25}} = \frac{\sqrt{\mathbf{3}}}{\mathbf{5}}$$

- **Simplify:** $\sqrt{\frac{25}{3}}$

  $$\sqrt{\frac{25}{3}} = \frac{\sqrt{25}}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

This looks very similar to the previous example, but this answer is not correct. Why? Because the denominator contains a radical, and the denominator must contain no radicals, or it is incorrect.

To get the correct answer, we must rationalize the denominator. That is, find some way to convert the fraction into a form where the denominator has only rational values (fractional or whole numbers).

Think back to elementary-school fractions – they could not be added unless they contained common denominators. To create these common denominators, the top and bottom would be multiplied by whatever the denominator needed. Since anything divided by itself is just 1, and multiplying by 1 doesn't change the value of whatever you're multiplying by the 1, multiplying by a strategic form of 1 can make the necessary computations possible, such as:

Use this same technique to rationalize radical denominators.

Look at the possibility of taking a 3 out of the denominator. This would be possible if there were two factors of 3 inside the radical. A pair of 3's can be created by multiplying both the top and bottom by a strategic form of 1 – root-three divided by root-three. This will not change the value, but will make simplification possible:

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3}\sqrt{3}} = \mathbf{\frac{5\sqrt{3}}{3}}$$

This last form, five, root-three, divided by three, is the correct answer.

- **Simplify:** $\mathbf{\frac{6\sqrt{2}}{\sqrt{3}}}$

$$\frac{6\sqrt{2}}{\sqrt{3}} = \frac{6\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{2}\sqrt{3}}{3} = \frac{6\sqrt{6}}{3} = \frac{3 \cdot 2\sqrt{6}}{3} = \frac{2\sqrt{6}}{1} = \mathbf{2\sqrt{6}}$$

Do not stop once you've rationalized the denominator. As the above demonstrates, you should always check to see if something remains to be simplified.

- **Simplify:** $\mathbf{\frac{3}{2+\sqrt{2}}}$

This expression is in the incorrect form since it has a radical in the denominator. However, multiplying the top and bottom by root-two will also not help because it does not clear the radical out of the denominator:

$$\left(2 + \sqrt{2}\right)\left(\sqrt{2}\right) = 2\sqrt{2} + 2$$

Multiplying through by another copy of the whole denominator will not help either:

$$\left(2 + \sqrt{2}\right)\left(2 + \sqrt{2}\right) = 4 + 4\sqrt{2} + 2 = 6 + \sqrt{2}$$

But look what happens when I multiply the denominator by the same numbers, but with the opposite sign in the middle – the conjugate:

$$(2+\sqrt{2})(2-\sqrt{2}) = 4 - 2\sqrt{2} + 2\sqrt{2} - 2 = 2$$

This multiplication made the radical terms cancel out. By using the conjugate, the necessary rationalization can be performed.

$$\frac{3}{2+\sqrt{2}} = \left(\frac{3}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right) = \frac{3(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} = \mathbf{\frac{6-3\sqrt{2}}{2}}$$

Nothing else will simplify, so we are finished.

- **Simplify:** $\mathbf{\frac{1+\sqrt{7}}{2-\sqrt{7}}}$

  Multiply the numerator and denominator by the conjugate of the denominator in order to simplify this expression. The denominator's multiplication results in the elimination of the radical:

$$\frac{1+\sqrt{7}}{2-\sqrt{7}} = \left(\frac{1+\sqrt{7}}{2-\sqrt{7}}\right)\left(\frac{2+\sqrt{7}}{2+\sqrt{7}}\right) = \frac{(1+\sqrt{7})(2+\sqrt{7})}{(2-\sqrt{7})(2+\sqrt{7})} = \frac{2+\sqrt{7}+2\sqrt{7}+7}{4+\sqrt{7}-\sqrt{7}-7} = \frac{9+3\sqrt{7}}{-3}$$
$$= \frac{3(3+\sqrt{7})}{-3} = -(3+\sqrt{7}) = \mathbf{-3-\sqrt{7}}$$

<u>Higher-Index Roots: Cubes, Fourth, and more</u>

Operations with cube roots, fourth roots, and other higher-index roots work similarly to square roots.

- **Simplify** $\mathbf{\sqrt[4]{16}}$

  In square roots, we can pull anything that we have two copies of. In fourth roots, we must find four copies of something to pull it out of the radical:

$$\sqrt[4]{16} = \sqrt[4]{2\times 2\times 2\times 2} = \mathbf{2}$$

Similarly, in a cube root any factor that occurs three times can be taken out; in a fourth root, take out any factor that occurs in fours; in a fifth root, take out any factor that occurs in fives; etc.

- **Simplify the cube root:** $\sqrt[3]{8}$

$$\sqrt[3]{8} = \sqrt[3]{\underline{2 \cdot 2 \cdot 2}} = \mathbf{2}$$

- **Simplify the cube root:** $\sqrt[3]{54}$

$$\begin{aligned} \sqrt[3]{54} &= \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 2} \\ &= \sqrt[3]{\underline{3 \cdot 3 \cdot 3}} \cdot \sqrt[3]{2} \\ &= \mathbf{3\sqrt[3]{2}} \end{aligned}$$

- **Simplify:** $\sqrt[3]{48}$

$$\sqrt[3]{48} = \sqrt[3]{3 \times 2 \times \underline{2 \times 2 \times 2}} = 2\sqrt[3]{3 \times 2} = \mathbf{2\sqrt[3]{6}}$$

- **Simplify:** $4\sqrt[3]{27}$

$$4\sqrt[3]{27} = 4\sqrt[3]{\underline{3 \times 3 \times 3}} = 4 \times 3 = \mathbf{12}$$

- **Simplify:** $\sqrt[5]{32x^{10}y^6z^7}$

$$\begin{aligned} &\sqrt[5]{32x^{10}y^6z^7} \\ &= \sqrt[5]{2 \times 2 \times 2 \times 2 \times 2 \times x^2x^2x^2x^2x^2 \times yyyyy \times y \times zzzzz \times z^2} \\ &= \mathbf{2x^2yz\sqrt[5]{yz^2}} \end{aligned}$$

<u>Manipulating Higher-Index Roots</u>

- **Simplify:** $\sqrt[3]{9}\ \sqrt[3]{24}$

$$\begin{aligned} \sqrt[3]{9}\ \sqrt[3]{24} &= \sqrt[3]{\underline{3 \times 3 \times 3} \times \underline{2 \times 2 \times 2}} \\ &= 3 \times 2 = \mathbf{6} \end{aligned}$$

- **Simplify:** $\sqrt[4]{75}\left(2\sqrt[4]{100}\right)$

$$\begin{aligned}\sqrt[4]{75}\left(2\sqrt[4]{100}\right) &= 2\sqrt[4]{3\times5\times5\times2\times2\times5\times5}\\ &= 2\sqrt[4]{\underline{5\times5\times5\times5}\times2\times2\times3}\\ &= 2\times5\sqrt[4]{2\times2\times3} = \mathbf{10\sqrt[4]{12}}\end{aligned}$$

- **Simplify:** $\sqrt[3]{8}+\sqrt[3]{64}$

$$\sqrt[3]{8}+\sqrt[3]{64} = \sqrt[3]{2\times2\times2}+\sqrt[3]{4\times4\times4} = 2+4 = \mathbf{6}$$

- **Simplify:** $\sqrt[3]{81}+5\sqrt[3]{3}$

$$\begin{aligned}\sqrt[3]{81}+5\sqrt[3]{3} &= \sqrt[3]{\underline{3\times3\times3}\times3}+5\sqrt[3]{3}\\ &= 3\sqrt[3]{3}+5\sqrt[3]{3} = \mathbf{8\sqrt[3]{3}}\end{aligned}$$

- **Simplify:** $\sqrt[3]{\frac{5}{27}}$

$$\sqrt[3]{\frac{5}{27}} = \frac{\sqrt[3]{5}}{\sqrt[3]{27}} = \frac{\sqrt[3]{5}}{\sqrt[3]{3\times3\times3}} = \mathbf{\frac{\sqrt[3]{5}}{3}}$$

- **Simplify:** $\sqrt[3]{\frac{27}{5}}$

This expression cannot be simplified as it is since it will have a radical in the denominator:

$$\sqrt[3]{\frac{27}{5}} = \frac{\sqrt[3]{3\times3\times3}}{\sqrt[3]{5}} = \frac{3}{\sqrt[3]{5}}$$

To rationalize a denominator containing a square root, we will need two copies of whatever factors are inside the radical. For a cube root, we need three copies. So, multiply that onto this fraction:

$$\left(\frac{3}{\sqrt[3]{5}}\right)\left(\frac{\sqrt[3]{5\times5}}{\sqrt[3]{5\times5}}\right)=\frac{3\sqrt[3]{25}}{\sqrt[3]{5\times5\times5}}=\mathbf{\frac{3\sqrt[3]{25}}{5}}$$

- **Simplify:** $\sqrt[4]{\frac{5}{72}}$

Since 72 = 8 × 9 = 2 × 2 × 2 × 3 ×3, I won't have enough of any of the denominator's factors to get rid of the radical. To simplify a fourth root, we need four copies of each factor. For this denominator's radical, we need two more 3s and one more 2:

$$\sqrt[4]{\frac{5}{72}}=\frac{\sqrt[4]{5}}{\sqrt[4]{72}}=\frac{\sqrt[4]{5}}{\sqrt[4]{2\times2\times2\times3\times3}}$$
$$=\left(\frac{\sqrt[4]{5}}{\sqrt[4]{2\times2\times2\times3\times3}}\right)\left(\frac{\sqrt[4]{2\times3\times3}}{\sqrt[4]{2\times3\times3}}\right)$$
$$=\frac{\sqrt[4]{5\times2\times3\times3}}{\sqrt[4]{\underline{2\times2\times2\times2}\times\underline{3\times3\times3\times3}}}$$
$$=\frac{\sqrt[4]{90}}{2\times3}=\mathbf{\frac{\sqrt[4]{90}}{6}}$$

<u>Radicals Expressed with Exponents</u>

Radicals can be expressed as fractional exponents. Whatever the index of the radical is becomes the denominator of the fractional power. For instance:

$$\sqrt{9}=\sqrt[2]{9}=9^{1/2}=3$$

For the square, or second root, we can write it as the one-half power, like this:

$\sqrt{2}=2^{1/2}$ or $\sqrt{4}=4^{1/2}=2$

The cube, or third root is the one-third power:

$$\sqrt[3]{8} = 8^{1/3} = 2$$

The fourth root is the one-fourth power:

$$\sqrt[4]{81} = 81^{1/4} = 3$$

The fifth root is the one-fifth power; and so on. This conversion process allows you to simplify some expressions that you might otherwise not have been able to.

- **Express $\sqrt[3]{2}\ \sqrt[4]{2}$ as a single radical term.**

  Convert the radicals to exponential expressions, and then apply the exponent rules to combine the factors:

  $$\sqrt[3]{2}\ \sqrt[4]{2} = 2^{1/3}\ 2^{1/4} = 2^{1/3+1/4} = 2^{7/12} = \sqrt[12]{2^7}$$

- **Simplify:** $\dfrac{\sqrt[3]{5}}{\sqrt{5}}$

  $$\frac{\sqrt[3]{5}}{\sqrt{5}} = \frac{5^{1/3}}{5^{1/2}} = 5^{1/3-1/2} = 5^{-1/6} = \frac{1}{5^{1/6}}$$

  $$= \frac{1}{\sqrt[6]{5}} = \left(\frac{1}{\sqrt[6]{5}}\right)\left(\frac{\sqrt[6]{5^5}}{\sqrt[6]{5^5}}\right)$$

  $$= \frac{\sqrt[6]{5^5}}{\sqrt[6]{5^6}} = \frac{\sqrt[6]{5^5}}{5}$$

- **Verify:** $\sqrt[3]{(2^3)} = 2$ and $\sqrt[4]{(3^4) = 3}$

Here are the steps used to complete these verifications:

$$\sqrt[3]{(2)^3} = \left(2^3\right)^{1/3} = \left(2^{3/1}\right)^{1/3} = 2^{\frac{3}{1}\bullet\frac{1}{3}} = 2^1 = 2$$

$$\sqrt[4]{(3)^4} = \left(3^4\right)^{1/4} = \left(3^{4/1}\right)^{1/4} = 3^{\frac{4}{1}\bullet\frac{1}{4}} = 3^1 = 3$$

Fractional exponents allow greater flexibility, are often easier to write than the equivalent radical format, and permit you to do calculations that you couldn't before. For instance:

$$\left(\sqrt[10]{25}\right)^5 = \left(25^{1/10}\right)^5 = 25^{\frac{1}{10}\cdot\frac{5}{1}} = 25^{1/2} = \sqrt{25} = 5$$

Whenever you see a fractional exponent, remember that the top number is the power, and the lower number is the root when converting back into the radical format.

$$\sqrt[m]{x^n} = (x^n)^{\frac{1}{m}} = \left(x^{\frac{1}{m}}\right)^n = x^{\frac{n}{m}}$$

Reference: http://www.purplemath.com/modules/exponent.htm