Pre-Calculus I

Fractions, Radicals, and Exponents

Parts of a Fraction

Numerator

From Latin "Numero" meaning number which gives the count of the number of parts.

Denominator

From Latin "Nomen" meaning name which gives the name of the part being counted.

Example

$\frac{8}{19}$ → Numerator
→ Denominator

Types of Fractions

Proper Fraction

Always less than 1, Example: $\frac{2}{5}$

Improper Fraction

Always greater than 1, Example: $\frac{7}{3}$

Mixed Number

Improper Fraction written as a sum of a Whole Number and a Proper Fraction,

Example: $3\frac{2}{7}$

Like Fractions

Has the same denominators, Example: $\frac{7}{12}$ and $\frac{5}{12}$

Unlike Fractions

Has different denominators, Example: $\frac{4}{7}$ and $\frac{6}{9}$

Conversions

Improper Fraction into a Mixed Number

Example: $\frac{23}{3}$

$\frac{23}{3} = 23 \div 3 = 7$ with remainder 2 $= 7\frac{2}{3}$

Mixed Number into an Improper Fraction

Example: $7\frac{5}{8}$

$7 + \frac{5}{8} = \frac{[(8*7)+5]}{8} = \frac{61}{8}$

Canceling

Example: $\frac{80}{150}$

$$\frac{80}{150}=\frac{2\times 2\times 2\times \cancel{2}\times 5}{5\times 5\times 3\times \cancel{2}}$$

$$\frac{80}{150}=\frac{2\times 2\times 2\times \cancel{2}\times \cancel{5}}{\cancel{5}\times 5\times 3\times \cancel{2}}=\frac{2\times 2\times 2}{5\times 3}=\frac{8}{15}$$

Reducing to Lowest Terms

Example: $\frac{54}{78}$

First, factor numerator and denominator.

$$\frac{54}{78}=\frac{2\times 3\times 3\times 3}{2\times 3\times 13}$$

Second, identify and eliminate the common factors

$$\frac{54}{78}=\frac{\cancel{2}\times\cancel{3}\times 3\times 3}{\cancel{2}\times\cancel{3}\times 13}=\frac{9}{13}$$

Multiplication

With Whole Numbers

Example: $3\times\frac{2}{7}=\frac{3\times 2}{7}=\frac{6}{7}$

Try This: $4\times\frac{3}{6}=?$

With Fractions

Example: $\frac{3}{5}\times\frac{2}{7}=\frac{3\times 2}{5\times 7}=\frac{6}{35}$

Try This: $\frac{2}{7}\times\frac{4}{5}=?$

Multiply and Reduce to Lowest Terms

Example: $\frac{5}{3}\times\frac{2}{5}=\frac{5\times 2}{3\times 5}=\frac{10}{15}=\frac{2}{3}$

Try This: $\frac{10}{4}\times\frac{3}{5}=?$

Division

Whole Number by a Fraction

Example: $\dfrac{10}{\frac{5}{2}} = 10 \times \dfrac{2}{5} = \dfrac{20}{5} = 4$

Note: Since we are dividing by a fraction, we are taking the reciprocal of the fraction and multiplying it with the number.

Try This: $\dfrac{6}{\frac{3}{2}} = ?$

Fraction by a Whole Number

Example: $\dfrac{\frac{10}{4}}{2} = \dfrac{10}{4} \times \dfrac{1}{2} = \dfrac{10}{8} = \dfrac{5}{4}$

Try This: $\dfrac{\frac{6}{5}}{10} = ?$

Fraction by a Fraction

Example: $\dfrac{\frac{10}{4}}{\frac{5}{2}} = \dfrac{10}{4} \times \dfrac{2}{5} = \dfrac{20}{20} = 1$

Try This: $\dfrac{\frac{6}{5}}{\frac{3}{2}} = ?$

Addition & Subtraction

With Same Denominator

Example: i. $\dfrac{2}{3} + \dfrac{4}{3} = \dfrac{6}{3} = 2$

ii. $\dfrac{7}{5} - \dfrac{2}{5} = \dfrac{5}{5} = 1$

Try This: $\dfrac{3}{5} + \dfrac{2}{5} = ?$

With Different Denominators

Addition and subtraction of 2 fractions with different denominators are done by using LCD (Least Common Denominator).

Example: i. $\dfrac{2}{3} + \dfrac{4}{6} = \dfrac{4+4}{6} = \dfrac{8}{6} = \dfrac{4}{3}$

ii. $\frac{8}{7}-\frac{1}{5}=\frac{40-7}{35}=\frac{33}{6}=\frac{11}{2}$

Try This: $\frac{5}{18}+\frac{5}{16}=?$

Radicals

Finding Roots

If $a^2 = 49$, then a=? To answer this question we need to find a number that can be multiplied by itself to result in the given number. That number is the **square root** of the given number.

To find the square root of 49, let us think of a number that can be multiplied by itself to give 49. The square root of 49 is 7, since 7 * 7 = 49. However, we get the same result if we multiply (-7) by (-7).

Thus, we can say that the positive square root of 49 is 7 written as $\sqrt{49} = 7$

The symbol $-\sqrt{\ }$ is used to denote negative square root.

Example: $-\sqrt{49} = -7$

Try This: $\sqrt{900} = ?$, $-\sqrt{1024} = ?$

The symbol $\sqrt{\ }$ is called a **radical sign** and, used alone, always represents the nonnegative square root. The number inside the radical sign is called the **radicand** and the entire expression, radical sign, and the radicand, is called a **radical**.

Products and Quotients of Radicals

Product Rule: $\sqrt{x}\cdot\sqrt{y}=\sqrt{x\cdot y}$

The above rule is true as long as x and y are not negative (that is the product of the two radicals is the radical of the product).

Example: $\sqrt{4}\cdot\sqrt{3} = \sqrt{4\cdot 3} = \sqrt{12}$

We can also simplify the radicals using the product rule.

Example: Simplify $\sqrt{12}$

$$\sqrt{12} = \sqrt{4\cdot 3}$$
$$= \sqrt{4}\cdot\sqrt{3}$$
$$= 2\sqrt{3} \quad (\text{since } \sqrt{4} = 2)$$

Try This: Simplify $\sqrt{72}$

Quotient Rule: $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$

This rule is true as long as x and y are not negative and y is not 0. That is, the quotient of the radicals is the radical of a quotient.

This can also be used to simplify radicals.

Example: Divide $\sqrt{15}$ by $\sqrt{5}$

$$\frac{\sqrt{15}}{\sqrt{5}} = \sqrt{\frac{15}{5}} = \sqrt{3}$$

Try This: Divide $\sqrt{81}$ by $\sqrt{9}$

Rationalizing the Denominator

Sometimes it is easier to find decimals if the denominators contain no radicals. Rationalizing is simply the process of eliminating a radical in the denominator.

Example: Rationalize the denominator for $\frac{9}{\sqrt{6}}$

Multiply both denominator and numerator by $\sqrt{6}$

$$\frac{9}{\sqrt{6}} = \frac{9 \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}} = \frac{9\sqrt{6}}{6} = \frac{3\sqrt{6}}{2}$$ (since $\sqrt{6} * \sqrt{6} = 6$)

Try This: Rationalize $\frac{12}{\sqrt{8}}$

Addition and Subtraction of Radicals

The distributive property can be used to add or subtract radicals.

Example: $2\sqrt{3} + 5\sqrt{3} = \sqrt{3}\ (2+5) = 7\sqrt{3}$ Distributive Property

Also, $11\sqrt{3} - 3\sqrt{3} = \sqrt{3}\ (11-3) = 8\sqrt{3}$

Only like radicals, those that are multiples of the same root of the same number, can be combined this way.

Example: $\sqrt{18} - \sqrt{27} = \sqrt{9 \cdot 2} - \sqrt{9 \cdot 3} = \sqrt{9} \cdot \sqrt{2} - \sqrt{9} \cdot \sqrt{3}$

$= 3\sqrt{2} - 3\sqrt{3}$

Since $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals, this difference cannot be simplified further

Try This: $= \sqrt{11} + \sqrt{11}$

Simplifying Radicals

1. If a radical is a perfect square, then the square roots should be used in place of the radical.

   Example: $\frac{\sqrt{121}}{\sqrt{9}} = \frac{11}{3}$

2. If a radical expression contains products of the radicals, the product rule for radicals should be used to get a single radical.

   Example: $\sqrt{4} \cdot \sqrt{3} = \sqrt{4 \cdot 3} = \sqrt{12}$

3. If a radicand has a factor that is a perfect square, the radical should be expressed as the product of the positive square root of the perfect square and the remaining radical factor.

   $\sqrt{75} = 5\sqrt{3}$

4. Any radical in the denominator should be changed to a rational number.

   Example: $\frac{12}{\sqrt{9}} = \frac{12 \cdot \sqrt{9}}{\sqrt{9} \cdot \sqrt{9}} = \frac{12\sqrt{9}}{9} = \frac{4\sqrt{9}}{3}$

5. If a radical expression contains sums or differences of radicals, the distributive property should be used to combine terms, if possible.

   Example: $2\sqrt{3} + 5\sqrt{3} = \sqrt{3}\ (2+5) = 7\sqrt{3}$

Solving Radical Equations

Example: Solve $\sqrt{21+x} = 3 + \sqrt{x}$

1. Arrange the terms so that there is no more than one radical on each side of the equation.
2. Square both sides,

   $(\sqrt{21+x})^2 = (3+\sqrt{x})^2$

   $21 + x = 9 + 6\sqrt{x} + x$

3. Combine terms and simplify

   $12 = 6\sqrt{x}$

   $2 = \sqrt{x}$

4. Square both sides a second time

   $4 = x$

5. Solve
   The solution is 4.
6. Check
   $\sqrt{21+x} = 3 + \sqrt{x}$

$\sqrt{21+4} = 3 + \sqrt{4}$
$\sqrt{25} = 3 + 2$
$5 = 5$
Hence the solution is correct.

Try This: $\sqrt{x} + 1 = 9x$

Exponents

Example: In the expression 5^2, 2 is the **exponent** and 5 is the **base**.

Rules of Exponents

1. For any positive integers m and n, $a^m \cdot a^n = a^{m+n}$

Example: $5^2 \cdot 5^3 = 5^{2+3} = 5^5$

2. For any real number a, $a^0 = 1$ (where a is a nonzero number)

Example: $(-5)^0 = 1$

3. For any real number a, and any integer n,

$$a^{-n} = \frac{1}{a^n} \text{ (where a is nonzero number)}$$

Example: $5^{-2} = \frac{1}{5^2}$

4. For any integers m and n, $(a^m)^n = a^{mn}$

Example: $(2^3)^5 = 2^{15}$

5. For any integer m, $(ab)^m = a^m b^m$

Example: $5(pq)^2 = 5(p^2 q^2) = 5p^2 q^2$

6. For any integer m, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example: $\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$

Try this: Solve $\left(\frac{(w^2)^5}{(q^5)^3}\right)$