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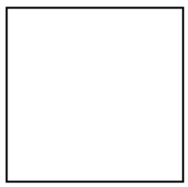
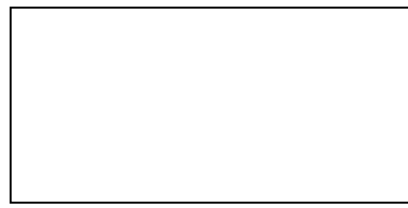
Geometrical Interpretation of Integrals

Suppose that we have to find the area of regular figures for example an square, we would directly use the formula Area of the square = (side).(side) for a rectangle we would use (length* width).



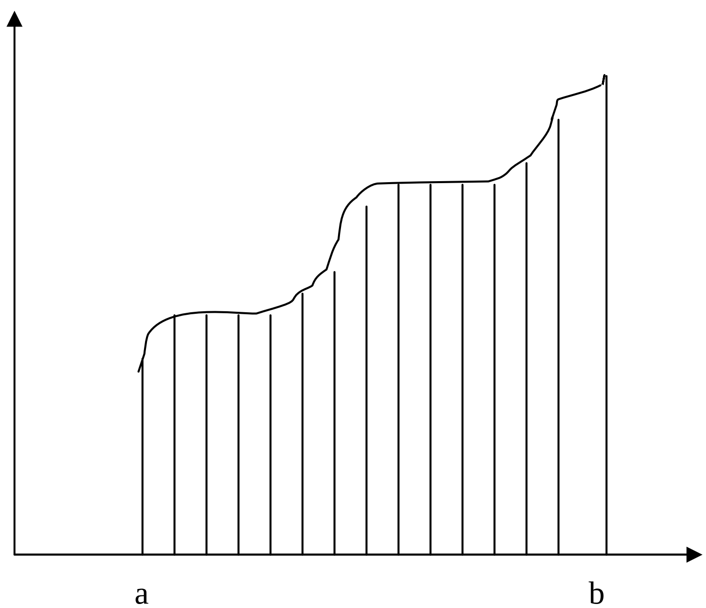


width

Side

Length

Now if we have to find the area of an irregular figure as shown in the figure below,



We divide the irregular figure into a number of small rectangles and we find out the areas of individual rectangles and the sum of the areas of all individual rectangles put together will give the area of the irregular figure.

Instead of this tedious process, we can **integrate** the function describing the irregular figure (curve) under the given interval [a, b] which is represented as

$$\int_a^b f(x)dx \text{ where } f(x) \text{ describes the irregular figure.}$$

“a” is called the lower limit and “b” the upper limit.

A parallel analogy to the concept of integration is Summation $\sum$. The only difference being summation is used when we talk about discrete quantities and integration comes into the picture while talking about continuous quantities.

Rules of Integration

Let f and g be functions and let a, b and c be constants, and assume that for each fact all the indicated definite integrals exist. Then the following are true:

Constants can be pulled out of integrals

$$\int cf(x)dx = c\int f(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

The integral of the sum of two functions equals the sum of the integrals of each function:

$$\int f(x)+g(x)dx = \int f(x)dx + \int g(x)dx$$

$$\int_a^b f(x)+g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

The integral of the difference of two functions equals the difference of the integrals of each function:

$$\int f(x)-g(x)dx = \int f(x)dx - \int g(x)dx$$

$$\int_a^b f(x)-g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

The integral from a to b of a function equals the integral from a to c plus the integral from c to b:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Basic Integrals

Integrals of Powers

$$\int kdx = kx + C$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Integrals of Exponentials

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln|a|} + C$$

Integrals of Trigonometric Functions

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

Integrals Involving Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

Techniques of Integration

Integration by Parts

Let $u=f(x)$ *and* $v=g(x)$ be functions of x, then

$$\int udv = uv - \int vdu$$

Ex *Find* $\int x \sin x dx$.

Let u=x and dv = sinx. Then du =1 and v = -cosx. Then use the formula

$$\begin{aligned}\int x \sin x dx &= uv - \int vdu \\ &= x(-\cos x) - \int (-\cos x) dx \\ &= -x\cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C.\end{aligned}$$

Try this: $\int t^2 e^t dt$

Solution: $t^2 e^t - 2te^t + 2e^t + C$

Trigonometric Integrals

Ex Evaluate $\int \cos^3 x dx$

$\cos^3 x = \cos^2 x . \cos x = (1 - \sin^2 x)\cos x$

We can evaluate the integral by substituting u=sinx so du=cosx dx

$$\int \cos^3 x dx = \int \cos^2 x . \cos x dx = \int (1 - \sin^2 x) \cos x dx$$

$$= \int (1-u^2)du = u - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

Try this: $\int \sin^5 x \cos^2 x dx$

Solution: $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

Trigonometric Substitution

Expression	*Substitution*	*Identity*
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$, $0 \le \theta < \pi/2$ Or $\pi \le \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

Ex Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}dx$

Let x=3 $\sin\theta$. Then

$$dx = 3\cos\theta d\theta$$

$$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2\theta d\theta = \int (\csc^2\theta - 1)d\theta = -\cot\theta - \theta + C$$

We know that $\sin\theta = \frac{x}{3}$, from that we can find the value of $\cot\theta = \frac{\sqrt{9-x^2}}{x}$

So $\frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(\frac{x}{3}) + C$.

Try this: $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

Solution: $-\frac{\sqrt{x^2+4}}{4x}+C$

Integration of Rational Functions by Partial Fractions:

Why do we have to split rational functions into partial fractions?
This would be clearer with an example. Suppose that we have to evaluate $\int \frac{2x+3}{x^2+3x+2}dx$ .It's tough to integrate this function as it is. So we split this into two partial fractions as $\int \frac{1}{x+2}dx$ and $\int \frac{1}{x+1}dx$. Now instead of $\int \frac{2x+3}{x^2+3x+2}dx$, we can integrate the individual integrals (which are much easier to integrate) and sum them up.

Two Types of Rational Functions:

Proper

- A rational function *f(x)=P(x)/Q(x)* where *P* & *Q* are polynomials is called proper if the degree of P is less than degree of Q.

Improper

- If f is improper, that is, deg(P) $\geq$ deg(Q) then we must divide Q into P until a remainder R(x) is obtained such that
  deg(P)< deg(Q).

Case I: *Q(x)* is a product of distinct linear factors

Ex Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

Step 1: If rational function is improper, divide the numerator with the denominator.
Step 2: Factor the denominator
We factor the denominator as

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

Step 3: Write the partial fraction decomposition of the integrand.
Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand has the form

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

Step 4:
To determine the values of A, B & C, we multiply both sides of the equation by the product of the denominators, x(2x-1)(x+2), obtaining

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

Simplify

$$x^2+2x-1 = (2A+B+2C)x^2+(3A+2B-C)x-2A$$

Step 5:
Equating the coefficients, we get
2A + B + 2C = 1
3A + 2B - C = 2
-2A = -1

By solving these equations, we get $A=\frac{1}{2}, B=\frac{1}{5}, C=\frac{-1}{10}$

Step 6: Integrate

So $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int (\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2})dx$

$$=\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K$$

Try this: $\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}dy$

Solution: $\frac{27}{5}\ln 2-\frac{9}{5}\ln 3$

Case II: <u>*Q(x)* is a product of linear factors, some of which are repeated.</u>

Ex Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

Step 1: Divide. The result of long division is

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

Step 2: Factor the denominator, we get

$$x^3-x^2-x+1=(x-1)(x^2-1)=(x-1)(x-1)(x+1)=(x-1)^2(x+1)$$

Step 3: Since the linear factor (x-1) occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

Step 4:
Multiplying by the least common denominator, we get

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$$=(A+C)x^2+(B-2C)x+(-A+B+C)$$

Step 5:
Equating the coefficients we get
A + C = 0
B - 2C = 4
-A + B + C = 0

By solving, we get A = 1, B = 2, C = -1 so,

Step 6:

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + K$$

Try this: $\int_0^1 \frac{2x}{(x+1)^2}dx$

Solution: $2\ln 2 + \frac{1}{2}$

- **Case III: *Q(x)* contains irreducible quadratic factors, none of which is repeated.**

If *Q(x)* has a factor $ax^2 + bx + c$ where $b^2 - 4ac < 0$ then the expression for *R(x)/Q(x)* will have a term of the form $\frac{Ax + B}{ax^2 + bx + c}$

Ex Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x}dx$.

Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we get

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$$

$$= (A + B)x^2 + Cx + 4A$$

Equating the coefficients, we obtain

A+B = 2, C = -1, 4A = 4

Thus A = 1, B = 1 and C = -1, so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2 + 4} \right) dx$$

In order to integrate the second term we split it into two parts:

$$\int \frac{x-1}{x^2 + 4}dx = \int \frac{x}{x^2 + 4}dx - \int \frac{1}{x^2 + 4}dx$$

We make the substitution $u = x^2 + 4$ in the first of these integrals so that $du = 2xdx$. We evaluate the second integral using the formula.

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}dx = \int \frac{1}{x}dx + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(x/2\right) + K$$

Try this: $\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3}dx$

Solution: $x + \frac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$

Case IV: **<u>Q(x) contains a repeated irreducible quadratic factor.</u>**

If *Q(x)* has the factor $(ax^2 + bx + c)^r$ where $b^2 - 4ac < 0$ then instead of a single partial fraction, the sum

$$\frac{A_1x + B_1}{(ax^2+bx+c)} + \frac{A_2x + B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_rx + B_r}{(ax^2+bx+c)^r}$$

occurs in the partial fraction decomposition of *R(x)/Q(x).*

Ex Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx$.

The form of the partial fraction decomposition is

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$-x^3 + 2x^2 - x + 1 = A(x^2+1)^2 + (Bx + C)x(x^2+1) + (Dx + E)x$$

$$= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex$$

$$-x^3 + 2x^2 - x + 1 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

Equating the coefficients and solving it , we get

A = 1, B = -1, C = -1, D = 1, E = 0,

So,

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx = \int\left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$$

$$= \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx - \int \frac{dx}{x^2+1} + \int \frac{xdx}{(x^2+1)^2}$$

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$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K$$

Try this: $\int \frac{x+4}{x^2+2x+5}dx$.

Solution: $\frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$

Reference
Stewart, James. *Calculus*
5th edition