Factoring a Polynomial Expression

Factoring a polynomial is expressing the polynomial as a product of two or more factors. Simply stated, it is somewhat the reverse process of multiplying.

To factor polynomials, we generally make use of the following properties or identities; along with other more sophisticated techniques.

Multiplied out form	Factored form	Property Name
$AB \pm AC$	$A(B \pm C)$	Distributive Property
$A^2 - B^2$	$(A+B)(A-B)$	Difference of two squares
$A^3 + B^3$	$(A+B)(A^2 - AB + B^2)$	Sum of two cubes
$A^3 - B^3$	$(A-B)(A^2 + AB + B^2)$	Difference of two cubes
$A^2 + 2AB + B^2$	$(A+B)^2$	Perfect Square trinomial
$A^2 - 2AB + B^2$	$(A-B)^2$	Perfect Square trinomial

Factorable Trinomials

A trinomial of the form $Ax^2 + Bx + C$, where *A*, *B*, and *C* are integers; *A*, *B*, and *C* have no common factors (are relatively prime); and $A \neq 0$, is said to be **factorable** over the integers if it can be expressed in the form $(Cx \pm D)(Ex \pm F)$, where *C*, *D*, *E*, and *F* are some integers.

A trinomial of the form $Ax^2 + Bx + C$, where *A*, *B*, and *C* are integers; *A*, *B*, and *C* have no common factors (are relatively prime); and $A \neq 0$, is said to be **prime** over the integers if it is not factorable over the integers.

Examples:

a) $3x^2 - 5x + 2$ factors as $(3x-2)(x-1)$ so $3x^2 - 5x + 2$ is a factorable trinomial.

b) $4x^2 + 12x + 9$ factors as $(2x+3)^2$ so $4x^2 + 12x + 9$ is a factorable trinomial.

c) $x^2 - x + 7$ is a prime trinomial as it does not factor over the integers.

Factoring Trinomials

A. Factoring Perfect Square Trinomials

A perfect square trinomial is a trinomial that can be identified as being of the form $A^2 \pm 2AB + B^2$. In other words, its first term is the square of some quantity, *A*, its last term is the square of some quantity, *B*, and its middle term is twice the product of *A* and *B*.

Whenever this is the case, one may apply the two perfect square identities:

$$A^2 + 2AB + B^2 = (A+B)^2$$

$$A^2 - 2AB + B^2 = (A-B)^2$$

to factor the trinomial.

Example 1: Factor $9x^2 - 12x + 4$

Solution: We observe that $9x^2 - 12x + 4$ can be written in the form $(3x)^2 - 2(3x)(2) + 2^2$. Since the first and the last term can be viewed as the squares of (3x) and (2), and disregarding its sign, the middle term is twice the product of (3x) and (2), $(3x)^2 - 2(3x)(2) + 2^2$ is of the form: $A^2 - 2AB + B^2$.

Now we will identify the variables as they relate to our problem:

$A = 3x$

$B = 2$

Substituting, we get $9x^2 - 12x + 4 = (3x)^2 - 2(3x)(2) + 2^2 = (2x-3)^2$

So, the factored form of the polynomial is $(2x-3)^2$

B. Factoring Trinomials of the form: $x^2 + bx + c$

A trinomial of the form $x^2 + bx + c$ is factorable over the integers, if one can find two numbers *p* and *q* such that $p \times q = c$ and $p + q = b$.

If two such numbers, *p* and *q*, exist, then the factored form of $x^2 + bx + c$ is $(x+p)(x+q)$.

Example 2: Factor $x^2 + 2x - 8$

Solution: First we need to find two numbers, *p* and *q*, whose product is -8 and whose sum is 2. We know that in order for a product to be negative, then one number must be negative and one must be positive. We get that $p = 4$ and $q = -2$.

Thus the factored form of $x^2 + 2x - 8$ is $(x+4)(x+(-2)) = (x+4)(x-2)$.

C. Factoring Trinomials of the form ax^2+bx+c where $a\neq 1$

There are two methods that we will discuss. They are Trial & Check Method and The ac-Method.

The Trial & Check Method

Step 1: List all possible pairs of factors whose product is the first term of the trinomial, namely ax^2.
Step 2: List all possible pairs of factors whose product is the last term of the trinomial, namely *c*.
Step 3: Try various combinations of these factors so that the product of the two first terms remains ax^2, the product of the two second terms remains equal to *c*, and the sum of the inner and outer products matches the middle term *bx*.

<u>Example 3:</u> Factor $6x^2-13x-8$ using the trial & check method.

<u>Solution:</u> Step 1: List all possible pairs of factors of $6x^2$: $6x\cdot 1x,\quad 2x\cdot 3x$

Step 2: List all possible pairs of factors of -8: $1\cdot -8,\quad -1\cdot 8,\quad 2\cdot -4,\quad -2\cdot 4$

Step 3: Write all possible product combinations that $6x^2-13x-8$ could be, using the factors of the first term and the third term. There are 16 possibilities. Continue until you have get the solution that you are looking for.

1) $(6x+1)(1x-8)$ Inner product: $1\cdot 1x=x$, Outer product: $6x\cdot -8=-48x$, Sum: $-47x$
2) $(6x-8)(1x+1)$ Inner product: $-8\cdot 1x=-8x$, Outer product: $6x\cdot 1=6x$, Sum: $-2x$
3) $(6x-1)(1x+8)$ Inner product: $-1\cdot 1x=-x$, Outer product: $6x\cdot 8=48x$, Sum: $47x$
4) $(6x+8)(1x-1)$ Inner product: $8\cdot 1x=8x$, Outer product: $6x\cdot -1=-6x$, Sum: $2x$
5) $(6x+2)(1x-4)$ Inner product: $2x\cdot 1=2x$, Outer product: $6x\cdot -4=-24x$, Sum: $-22x$
6) $(6x-4)(1x+2)$ Inner Product: $-4\cdot 1x=-4x$, Outer product: $6x\cdot 2=12x$, Sum: $8x$
7) $(6x-2)(1x+4)$ Inner Product: $-2\cdot 1x=-2x$, Outer product: $6x\cdot 4=24x$, Sum: $22x$
8) $(3x-8)(2x+1)$ Inner Product: $-8\cdot 2x=-16x$, Outer product: $3x\cdot 1=3x$, Sum: $-13x$

Since we found the correct middle term (sum), we can stop. Thus, the factored form is

$$(3x-8)(2x+1)$$

The ac-Method

Step 1: Find two numbers, *p* and *q*, satisfying the two properties: $pq=ac$ and $p+q=b$
Step 2: Write the trinomial ax^2+bx+c as a four term polynomial: $ax^2+px+qx+c$
Step 3: Factor the four term polynomial using grouping: $(ax^2+px)+(qx+c)$

Example 4: Factor $6x^2 - 13x - 8$ using the ac-method

Solution: Since *ac* is negative, it indicates that *p* and *q* must have opposite sign. Without loss of generality, we can assume that *p* is the positive number and *q* is the negative number.
We need to find two numbers that multiply to give -48 and add to give -13. Make a chart to help find the exact *p* and *q* we need. (*p* is all possible factors of 48)

p	*q*	*pq=-48*	*p+q=-13*
1	-48	-48	-47
2	-24	-48	-22
3	-16	-48	-13
4			
6			
8			
12			
16			
24			
48			

Stop! We found the correct pair!

So, $p = 3$ and $q = -16$. Our trinomial now becomes $6x^2 + 3x - 16x - 8$

$$\begin{aligned} 6x^2 - 13x - 8 &= (6x^2 + 3x) + (-16x - 8) \\ &= 3x(2x+1) - 8(2x+1) \\ &= (2x+1)(3x-8) \end{aligned}$$

Factoring the Difference of Two Squares

To factor a difference of two perfect squares, identify the two perfect squares and apply the identity: $A^2 - B^2 = (A+B)(A-B)$.

Example 5: Factor $16x^2 - 25y^2$

Solution: $16x^2 - 25y^2 = (4x)^2 - (5y)^2$
So, $A = 4x$ and $B = 5y$. Now, just plug into the identity and we get

$$16x^2 - 25y^2 = (4x+5y)(4x-5y)$$

Factoring Sums/Differences of Two Cubes

To factor the sum or difference of two perfect cubes, identify the two perfect cubes and apply the appropriate identity:

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Example 6: Factor $64x^3 - 125y^3$

Solution: $64x^3 - 125y^3 = (4x)^3 - (5y)^3$
So, $A = 4x$ and $B = 5y$. Thus, by the difference identity, we get

$$\begin{aligned} 64x^3 - 125y^3 &= (4x - 5y)[(4x)^2 + (4x)(5y) + (5y)^2] \\ &= (4x - 5y)(16x^2 + 20xy + 25y^2) \end{aligned}$$

Factoring Chart