Differentiation

The Derivative

The slope of the tangent line to a graph *f* at *x* gives you the derivative of the function *f*. The **derivative of *f* at *x*** is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided this limit exists. A function is **differentiable** at *x* if its derivative exists at *x*. The process of finding derivatives is called **differentiation**. Other notations to denote the derivative of *y = f(x)* are as follows:

$$\frac{dy}{dx}, y', \frac{d}{dx}\left[f(x)\right] \text{ and } D_x\left[y\right]$$

Example: Find the derivative of $f(x)=3x^2 - 2x$.

Solution: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{[3(x+\Delta x)^2 - 2(x+\Delta x)] - [3x^2 - 2x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3(\Delta x)^2 - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2 - 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(6x + 3\Delta x - 2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (6x + 3\Delta x - 2)$$

$$\Rightarrow f'(x) = 6x - 2$$

Try This: Find the derivative of $y = \frac{2}{t}$

Rules of Differentiation

The Constant Rule

The derivative of a constant function is zero. That is,

$$\frac{d}{dx}[c] = 0, \ c \text{ is a constant.}$$

Example: Find the derivative of $f(x) = -\frac{4}{3}$.

Solution: $f'(x) = 0$

Try This: $f(x) = -5.7345$

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The Simple Power Rule

$\frac{d}{dx}[x^n] = nx^{n-1}$, n is any real number.

Example: Find the derivative of $f(x) = 3x^2$.
Solution: $f'(x) = 3(2.x^1) = 6x$

Try This: Find the derivative of $y = \frac{1}{x^3}$

The Constant Multiple Rule

If f is a differentiable function of x, and c is a real number, then

$\frac{d}{dx}[cf(x)] = cf'(x)$, c is a constant.

Example: Find the derivative of $f(x) = \frac{4t^2}{5}$.

Solution: $f(t) = \frac{4}{5}t^2$

$$f'(t) = \frac{d}{dt}\left[\frac{4}{5}t^2\right]$$

$$f'(t) = \frac{4}{5}\left[\frac{d}{dt}\left(t^2\right)\right]$$

$$f'(t) = \frac{4}{5}(2t)$$

$$f'(t) = \frac{8}{5}t$$

Try This: Find the derivative of $y = 3x^4$.

General Formulas for Exponential, Logarithmic & Trigonometric Functions

$\frac{d}{dx}(e^x) = e^x$ $\qquad$ $\frac{d}{dx}(a^x) = a^x \ln a$

$\frac{d}{dx}\ln|x| = \frac{1}{x}$ $\qquad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\cos x) = -\sin x$ $\qquad$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\qquad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

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The Sum and Difference Rule

The derivative of the sum (or difference) of two differentiable functions is the sum (or difference) of their derivatives.

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x) \qquad \text{Difference Rule}$$

Example: Find the derivative of $f(x) = -\frac{1}{2}x^4 + 3x^3 - 2x - \sin\theta$.

Solution: $f'(x) = -\frac{1}{2}(4x^{3-1}) + 3(3x^{2-1}) - 2(x^{1-1}) - \cos\theta$

$f'(x) = -2x^2 + 9x^2 - 2 - \cos\theta$

Try This: Find the derivative of $f(x) = x^3 - 4x + 2$

The Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x).g(x)] = f(x)g'(x) + g(x)f'(x)$$

Example: Find the derivative of $y = x^2 \sin x$.

Solution: $\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2)$

$\frac{dy}{dx} = x^2\cos x + 2x\sin x$

Try This: Find the derivative of $y = \left(\frac{1}{x}+1\right)(x-1)$

The Quotient Rule

The derivative of the quotient of two differentiable functions is equal to the denominator times the derivative the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Example: Find the derivative of $y = \frac{x-1}{2x+3}$

Solution: $\frac{dy}{dx} = \frac{(2x+3)\frac{d}{dx}[x-1] + (x-1)\frac{d}{dx}[2x+3]}{(2x+3)^2}$

$$\frac{dy}{dx} = \frac{(2x+3)(1) - (x-1)(2)}{(2x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2x+3) - (2x-2)}{(2x+3)^2}$$

$$\frac{dy}{dx} = \frac{2x+3-2x+2}{(2x+3)^2}$$

$$\frac{dy}{dx} = \frac{5}{(2x+3)^2}$$

Try This: Find the derivative of $y = \frac{\sec x}{1+\tan x}$

The General Power Rule

If $y = [u(x)]^n$, where *u* is a differentiable function of *x* and *n* is a real number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

Example: Find the derivative of $f(x) = (3x-2x^2)^3$

Solution: The inside function is $u = 3x - 2x^2$

$$f'(x) = 3(3x-2x^2)^2\frac{d}{dx}[3x-2x^2]$$

$$f'(x) = 3(3x-2x^2)^2(3-4x)$$

$$f'(x) = (9-12x)(3x-2x^2)^2$$

Try This: Find the derivative $y = \sqrt[3]{(x^2+2)}$

The Chain Rule

If *y = f(u)* is a differentiable function of *u*, and *u = g(x)* is a differentiable function of *x*, then *y = f(g(x))* is a differentiable function of *x*, and

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

or, equivalently,

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$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

Example: Find the derivative of $y = \ln\ (2 + \sin x)$

Solution: Inside function $u = 2 + \sin x$

$$\frac{du}{dx} = \cos x$$

Original Function $y = \ln u$

$$\frac{dy}{du} = \frac{1}{u}$$

Substituting for *u*, we get

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{(2 + \sin x)}$$

Plugging in the chain rule, we get,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{(2 + \sin x)}.\cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{(2 + \sin x)}$$

Try This: Find the derivative of $y = \dfrac{1}{x - 1}$

Higher – Order Derivatives

The derivative of f' is the **second derivative** of f and is denoted by f''

$$\frac{d}{dx}[f'(x)] = f''(x) \qquad \textit{Second Derivative}$$

The derivative of f'' is the **third derivative** of f and is denoted by f'''

$$\frac{d}{dx}[f''(x)] = f'''(x) \qquad \textit{Third Derivative}$$

By continuing this process, you obtain higher-order derivatives of *f*.

Example: Find the higher – order derivatives of $f(x) = 2x^4 - 3x^2 + xe^x$

Solution: $f'(x) = 8x^3 - 6x + xe^x + e^x$ *First Derivative*

$f''(x) = 24x^2 - 6 + xe^x + e^x + e^x$ *Second Derivative*

$f'''(x) = 48x + xe^x + 3e^x$ *Third Derivative*

$f^{(4)}(x) = 48 + xe^x + 4e^x$ *Fourth Derivative*

Try This: Find the higher – order derivatives of $g(x) = -x^4 + 2x^3 + x + 4$

Implicit Differentiation

Consider an equation involving *x* and *y* in which *y* is a differentiable function of *x*. You can use the following steps to find *dy/dx*.

1. Differentiate both sides of the equation *with respect to x.*
2. Write the result so that all terms involving *dy/dx* are on the left side of the equation and all other terms are on the right side of the equation.
3. Factor *dy/dx* out of the terms on the left side of the equation.
4. Solve for *dy/dx* by dividing both sides of the equation by the left-hand factor that does not contain *dy/dx.*

Example: Find the *dy/dx* for the equation $x^2 + 4y^2 = 4$

Solution: $x^2 + 4y^2 = 4$

1. $\frac{d}{dx}[x^2 + 4y^2] = \frac{d}{dx}[4] \Rightarrow 2x + 8y\left(\frac{dy}{dx}\right) = 0$
2. $8y\left(\frac{dy}{dx}\right) = -2x$
3. $\left(\frac{dy}{dx}\right) = \frac{-2x}{8y}$
4. $\frac{dy}{dx} = -\frac{x}{4y}$

Try This: Find the *dy/dx* for the equation $y^3 + y^2 - 5y - x^2 = -4$.

Reference

Larson, Roland, Hostetler, Robert, and Edwards, Bruce. *Brief Calculus with Applications*. 4th ed. Lexington, MA: D.C. Heath and Co, 1995. 81-155.