Applications of Differentiation

(217) 206 - 6503 CTL ctl@uis.edu

Maximum and Minimum Values

A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D.

Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D and number $f(c)$ is called the **minimum value** of f on D.

The maximum and minimum values of f are called the **extreme values** of f.

A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. [This means that $f(c) \geq f(x)$ for all x in some open interval containing c.] Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c.

Example 1 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x. Likewise, $\cos(2n+1)\pi = -1$ is its minimum value, where n is any integer.

Example 2 The graph of the function $f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leq x \leq 4$ is shown. We can see that $f(1) = 5$ is the local maximum, whereas the absolute maximum is $f(-1) = 37$. This absolute maximum is not a local maximum because it occurs at an endpoint. Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor absolute maximum at $x = 4$.

The Extreme Value Theorem
If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$.

Fermat's Theorem
If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Critical Number
A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example Find the critical numbers of $f(x) = x^{3/5}(4-x)$.
Solution The Product Rule gives

$$f'(x) = \frac{3}{5}x^{-2/5}(4-x) + x^{3/5}(-1) = \frac{3(4-x)}{5x^{2/5}} - x^{3/5}$$

$$= \frac{3(4-x)-5x}{5x^{2/5}} = \frac{12-8x}{5x^{2/5}}$$

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Therefore, $f^{'}(x)=0$ if $12-8x=0$, that is, $x=\frac{3}{2}$, and $f^{'}(x)$ does not exist when $x=0$.

Thus the critical numbers are $\frac{3}{2}$ and 0.

If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

1. Find the values of f at the critical numbers of f in (a,b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example Find the absolute maximum and minimum values of the function

$$f(x)=x^3-3x^2+1 \qquad -\frac{1}{2}\le x\le 4$$

Solution Since f is continuous on $\left[-\frac{1}{2},4\right]$, we can use the Closed Interval Method:

$$f(x)=x^3-3x^2+1$$
$$f^{'}(x)=3x^2-6x=3x(x-2)$$

Since $f^{'}(x)$ exists for all, the only critical numbers of f occur when $f^{'}(x)=0$, that is, $x=0$ or $x=2$.

The values of f at these critical numbers are

$$f(0)=1 \qquad f(2)=-3$$

The values of f at the endpoints of the interval are

$$f(-\frac{1}{2})=\frac{1}{8} \qquad f(4)=17$$

Comparing the four numbers, we see that absolute maximum value is $f(4)=17$ and the absolute minimum value is $f(2)=-3$.

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How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

a) If $f^{'}(x) > 0$ on an interval, then f is increasing on that interval.
b) If $f^{'}(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test

Suppose that c is a critical number of a continuous function f.

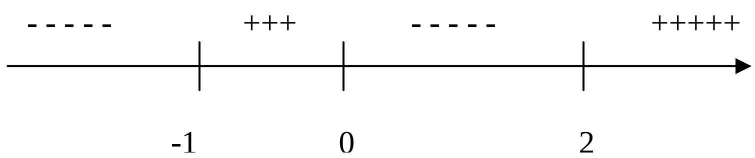
a) If $f^{'}$ changes from positive to negative at c, then f has a local maximum at c.
b) If $f^{'}$ changes from negative to positive at c, then f has a local minimum at c.
c) If $f^{'}$ does not change sign at c, then f has no local maximum or minimum at c.

Example Find the local maximum and minimum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution: $f^{'}(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$

To use the I/D test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the three factors of $f'(x)$, namely, $12x$, $x-2$ and $x+1$. We divide the real line into intervals whose endpoints are the critical numbers -1, 0 and 2.

The sign of $f'(x)$ is represented on the number line.



We see that $f'(x)$ changes from negative to positive at -1, so $f(-1) = 0$ is a local minimum value by the First Derivative Test. Similarly, f' changes from negative to positive at 2, so $f(2) = -27$ is also a local minimum value. $f(0) = 5$ is a local maximum value because $f'(x)$ changes from positive to negative at 0.

Definition

If the graph of f lies above all of its tangents on an interval *I*, then it is called **concave upward** on *I*. If the graph of f lies below all of its tangents on *I*, it is called **concave downward** on *I*.

Concavity Test

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on *I*.

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on *I*.

Definition

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

Suppose f'' is continuous near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

c) If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.

Example Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution: If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

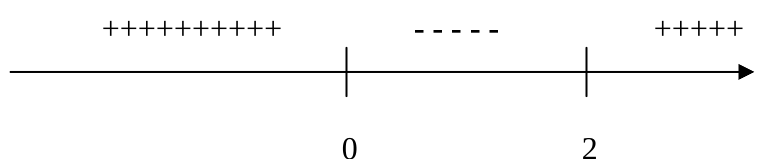
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$. $f(3) = -27$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints.
The sign of $f''(x)$ is represented on the number line.



The point $(0,0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also (2, -16) is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve.

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Optimization Problems

Steps in Solving Optimization Problems

1. Understand the problem.
2. Draw a Diagram.
3. Introduce Notation. Assign a symbol to the quantity that is to be maximized or minimized (say Q). Also select symbols (a, b, c, …, x, y) for other unknown quantities and label the diagram with these symbols.
4. Express Q in terms of some of the other symbols from Step 3.
5. Find the relationship between Q and the unknown quantities.
6. Find the absolute maximum or minimum value of f .

Example A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution Draw the diagram as in the figure 1 below, where r is the radius and h is the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides).

From figure 2, we see that that the sides are made from a rectangular sheet with dimensions $2\pi\ r$ and h. So the surface area is

$S = 2\pi\ r^2 + 2\pi\ rh$ since the area of two circles for top and bottom gives us $2\pi\ r^2$ and surface area of the rectangular sheet is $2\pi\ rh$.

To eliminate h we use the fact the volume is given as 1 L, which we take to be $1000\,cm^3$

Volume of a cylinder is $\pi\ r^2\ h$, so

$$\pi\ r^2\ h = 1000$$

We only want one variable in the equation instead of two, so solve for one of the variables and substitute…

Which gives $h = 1000/\pi r^2$. Substitution of this into the expression of S gives

$$S = 2\pi\ r^2 + 2\pi\ r\left(\frac{1000}{\pi r^2}\right) = 2\pi\ r^2 + \frac{2000}{r}.$$

Therefore, the function that we want to minimize is

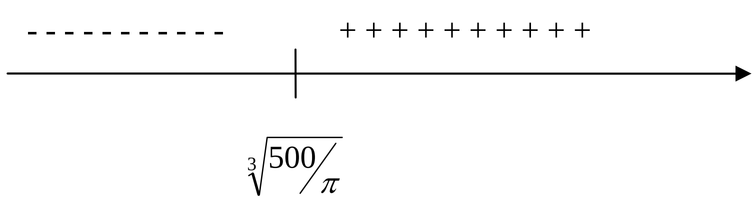
$$S(r) = 2\pi\ r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$S'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $S'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi} \cong 5.4192$

The sign of $S'(r)$ is represented on the number line.



So S is decreasing for all r to the left of the critical number and increasing for all r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an absolute minimum.

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius.

Reference:

Stewart, James. *Calculus* 5th edition